CHALMERS UNIVERSITY OF TECHNOLOGY
Department of Signals and Systems
Control, Automation and Mechatronics

SSY280 Model Predictive Control
Final exam 2012-03-08

M 14.00 – 18.00

Teacher: Bo Egardt, tel 3721.

The following items are allowed to bring to the exam:

- Chalmers approved calculator.
- One A4 page with your own notes.
- Beta.

Grading: The exam consists of 5 problems of in total 30 points. The nominal grading is 12 (3), 18 (4) and 24 (5). Solutions may be short, but should always be clear and **well motivated**!
Grading results are posted not later than March 22 at the billboard on the 5th floor. Review of the grading is offered on March 22 at 12.30 – 13.30. If you cannot attend at this occasion, any objections concerning the grading must be filed in written form not later than two weeks after the regular review occasion.

Note that solutions should be given in English!

GOOD LUCK!

Problem 1.

a. The standard plant model used in the course is given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= C_y x(k) \\ z(k) &= C_z x(k) \end{aligned}$$

Show how this model can be transformed into a model that instead uses *control moves* Δu as input. (2 p)

b. Explain why the type of MPC algorithms studied in the course give *convex* optimization problems to solve. (2 p)

c. What is meant by *functional predictive control*? (2 p)

d. In some cases, it can be motivated to define setpoints not only for outputs, but also for inputs. *When* is this the case, and *why* is it done? (2 p)

e. Using state constraints (as opposed to control constraints) in MPC has a drawback — which one?

(2 p)

Solution:

a.

$$\begin{aligned} \xi^+ &= \mathcal{A}\xi + \mathcal{B}\Delta u \\ y &= \mathcal{C}x \end{aligned}$$

with

$$\mathcal{A} = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \quad \mathcal{B} = \begin{bmatrix} B \\ I \end{bmatrix} \quad \mathcal{C} = \begin{bmatrix} C & 0 \end{bmatrix}$$

b. The optimization problem is convex since 1) the objective function is quadratic, hence convex, and 2) the constraints are affine, i.e. the feasible set is described as the solution set of a number of linear (in)equalities, which is convex.

c. Functional predictive control makes use of a parametrization of the sequence of future control outputs as a weighted sum of basis functions, often polynomials as function of time.

d. *Setpoints for inputs are used when there are more control inputs than controlled outputs with setpoints. It is done to avoid multiple solutions to the setpoint tracking problem.*

e. *State constraints imply that there is a risk of infeasibility of the optimization problem.*

Problem 2.
Consider the first order system described by

$$y(k+1) = ay(k) + u(k)$$

You are supposed to work out the details for a simple MPC controller for this system. The prediction horizon is 2 and the control horizon is 1. The controller is based on minimization of the objective

$$V_2(y(k), u(k|k)) = (\hat{y}(k+1|k) - r(k))^2 + \alpha(\hat{y}(k+2|k) - r(k))^2$$

where α is a tuning parameter.

a. Solve the optimization problem and give an expression for the control law. *Hint.* Use standard assumptions. (3 p)

b. Determine the closed-loop poles in the two extreme cases with $\alpha = 0$ and $\alpha \to \infty$, respectively. (2 p)

Solution:

a. *The predictions are given by*

$$\begin{aligned}\hat{y}(k+1|k) &= ay(k) + u(k|k)\\ \hat{y}(k+2|k) &= a\hat{y}(k+1|k) + \hat{u}(k+1|k) = a^2y(k) + (1+a)\hat{u}(k|k)\end{aligned}$$

where the standard assumption $\hat{u}(k+1|k) = \hat{u}(k|k)$ has been used (since the control horizon is 1). To find the minimizing control, differentiate w.r.t. $\hat{u}(k|k)$ and set the result equal to zero, giving (use the abbreviations $u = \hat{u}(k|k)$, $y = y(k)$, $r = r(k)$):

$$\begin{aligned}2(ay + u - r) + 2\alpha(1+a)\big(a^2y + (1+a)u - r\big) = 0 \quad &\Leftrightarrow\\ a\big(1 + a\alpha(1+a)\big)y + \big(1 + \alpha(1+a)^2\big)u - \big(1 + \alpha(1+a)\big)r = 0&\end{aligned}$$

which results in the control law

$$u(k) = \frac{1+\alpha(1+a)}{1+\alpha(1+a)^2}r(k) - \frac{a(1+a\alpha(1+a))}{1+\alpha(1+a)^2}y(k)$$

b. The control law in the extreme cases become

$$u(k) = \begin{cases} r(k) - ay(k), & \alpha = 0 \\ \frac{1}{1+a}r(k) - \frac{a^2}{1+a}y(k), & \alpha \to \infty \end{cases}$$

which gives the closed-loop dynamics

$$y(k+1) = \begin{cases} r(k), & \alpha = 0 \\ \frac{a}{1+a}y(k) + \frac{1}{1+a}r(k), & \alpha \to \infty \end{cases}$$

The closed-loop poles are thus $p = 0$ *and* $p = a/(1+a)$ *in the two cases.*

Problem 3.
You are presented with the task to design an MPC controller (of the type we have studied in the course) for a system, which can be described by the state space model

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ y(k) &= Cx(k) \end{aligned} \tag{1}$$

with the following characteristics:

- The system is strictly stable, i.e. the eigenvalues of A are strictly inside the unit disc.
- There are 2 control inputs.
- There are 3 outputs, all of which are measured, but only outputs no 1 and 3 are controlled, with r_1 and r_3 as setpoints.
- The controlled outputs are subject to constant but unknown additive output disturbances.

a. Show how the two disturbance terms can be estimated using a simple observer, whose error dynamics is determined by eigenvalues in the origin and eigenvalues of A.
Hint. Use a modified DMC scheme. (2 p)

b. Based on the estimates, determine suitable steady state targets, without considering possible constraints. (2 p)

c. Considering that the first and third outputs are controlled, we choose the weighting matrix Q for the state-dependent part of the stage cost in the optimization objective as

$$Q = c_1 c_1^T + c_3 c_3^T,$$

where c_1^T and c_3^T are the first and third row of the matrix C, i.e

$$C = \begin{bmatrix} c_1^T \\ c_2^T \\ c_3^T \end{bmatrix}$$

Give the full expression for the state dependent part of the objective. (2 p)

Solution:

a. *Using the model*

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ d(k+1) &= d(k) \\ y(k) &= Cx(k) + C_d d(k) \end{aligned}$$

with $d^T = [d_1 \; d_2]$ *and*

$$C_d = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

an observer of DMC type is given by

$$\begin{aligned} \hat{x}^+ &= A\hat{x} + Bu \\ \hat{d}^+ &= \hat{d} + H(y - C\hat{x} - C_d\hat{d}) \end{aligned}$$

with

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b. *Since there are two inputs and two controlled outputs, the steady state target can be determined from the solution to*

$$\begin{bmatrix} I - A & -B \\ HC & 0 \end{bmatrix} \begin{bmatrix} x_s \\ u_s \end{bmatrix} = \begin{bmatrix} 0 \\ r - \hat{d} \end{bmatrix}$$

where $r^T = [r_1 \; r_3]$.

c. *From above, we have* $HCx_s = r - \hat{d}$, *which can be written as*

$$\begin{aligned} c_1^T x_s &= r_1 - \hat{d}_1 \\ c_3^T x_s &= r_3 - \hat{d}_2 \end{aligned}$$

The state dependent part of the stage cost can then be written

$$\sum_{i=0}^{N-1} (\hat{x}(k+i|k) - x_s)^T Q (\hat{x}(k+i|k) - x_s)$$
$$= \sum_{i=0}^{N-1} \left[\left(c_1^T \hat{x}(k+i|k) + \hat{d}_1(k) - r_1(k) \right)^2 + \left(c_3^T \hat{x}(k+i|k) + \hat{d}_2(k) - r_3(k) \right)^2 \right]$$

Problem 4.
The prediction and control horizons are two of the design parameters for an MPC algorithm.

a. What is a reasonable choice of prediction horizon? (1 p)

b. What is the reason why the control horizon is usually chosen smaller than the prediction horizon? (1 p)

c. Show in detail how equality constraints, representing the state equations of the model
$$x^+ = Ax + Bu,$$
can be formed for the case when the prediction and control horizon are *not equal*. Which assumption do you need? (3 p)

Solution:

- *Approximately equal to the settling time of the system.*
- *The control horizon strongly influences the computational complexity.*
- *Done in assignment 2.*

Problem 5.

a. When proving stability of a model predictive controller, a fundamental step is the following assumption (taken from the lecture notes):
$$\min_{u \in \mathbb{U}} \{ V_f(f(x,u)) + l(x,u) \mid f(x,u) \in \mathbb{X}_f \} \leq V_f(x), \quad \forall x \in \mathbb{X}_f$$
Explain how this assumption can be shown to hold by choosing $\mathbb{X}_f = \{0\}$. (1 p)

b. During the lectures, a simple MPC with $N = 2$ was studied for an integrator process $x^+ = x + u$ with control constraint $-1 \leq u \leq 1$. It was shown, using geometric arguments, that the resulting control law could be described as a saturated linear feedback:

$$u(x) = \begin{cases} 1 & x \leq -5/3 \\ -3/5 \cdot x & -5/3 \leq x \leq 5/3 \\ -1 & x \geq 5/3 \end{cases}$$

Based on the expression for the objective,

$$V_N(x, \mathcal{U}) = \mathcal{U}^T H \mathcal{U} + 2 \cdot [2x \quad x]\mathcal{U} + 3x^2,$$

where $x = x(0)$, $\mathcal{U} = [u(0) \quad u(1)]^T$ and $H = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$, verify that the control law obtained satisfies the KKT conditions. What happens when $x = -3$ and $x = 3$, respectively?

Hint. The KKT conditions for this case (i.e. no equality constraints) are given by (note that here x is a general notation for the vector of decision variables) are:

(i) Primal constraints: $g_i(x) \leq 0,\ i = 1, \ldots, m$

(ii) Dual constraints: $\lambda_i \geq 0,\ i = 1, \ldots, m$

(iii) Complementary slackness: $\lambda_i g_i(x) = 0,\ i = 1, \ldots, m$

(iv) Gradient of the Lagrangian equal to zero:

$$\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) = 0$$

(3 p)

Solution:

a. Trivially fulfilled if $f(0,0) = 0$, $V_f(0) = 0$ *and* $l(0,0) = 0$.

b. The KKT conditions become

$$\begin{aligned} g_1 &= u(0) - 1 \le 0 \\ g_2 &= -u(0) - 1 \le 0 \\ g_3 &= u(1) - 1 \le 0 \\ g_4 &= -u(1) - 1 \le 0 \\ \lambda &\ge 0 \\ \lambda_i g_i &= 0 \\ H\mathcal{U} + \begin{bmatrix} 2x \\ x \end{bmatrix} &+ \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \lambda = 0 \end{aligned}$$

Note that λ_1, λ_2 can not be nonzero simultaneously; the same holds for λ_3, λ_4. It can then be verified that the solution is obtained from 5 cases:

(i) $\lambda = 0$ *(no active constraints):*

$$H\mathcal{U} + \begin{bmatrix} 2x \\ x \end{bmatrix} = 0 \Rightarrow \mathcal{U} = \begin{bmatrix} -3/5 \\ -1/5 \end{bmatrix} x, \quad -5/3 \le x \le 5/3$$

(ii) $\lambda_1 > 0\,(\Rightarrow \lambda_2 = 0, g_1 = 0), \lambda_3 = \lambda_4 = 0$*:*

$$u(0) = 1,\ u(1) = -\frac{1+x}{2}, \quad -3 \le x \le -5/3$$

(iii) $\lambda_1 > 0\,(\Rightarrow \lambda_2 = 0, g_1 = 0), \lambda_3 > 0 (\Rightarrow \lambda_4 = 0, g_3 = 0)$*:*

$$u(0) = 1,\ u(1) = 1, \quad x \le -3$$

(iv) $\lambda_2 > 0\,(\Rightarrow \lambda_1 = 0, g_2 = 0), \lambda_3 = \lambda_4 = 0$*:*

$$u(0) = -1,\ u(1) = \frac{1-x}{2}, \quad 5/3 \le x \le 3$$

(v) $\lambda_2 > 0\,(\Rightarrow \lambda_1 = 0, g_2 = 0), \lambda_4 > 0 (\Rightarrow \lambda_3 = 0, g_4 = 0)$*:*

$$u(0) = -1,\ u(1) = -1, \quad 3 \le x$$

As seen from the above, $u(1)$ leaves/enters the constraint at $x = -3$ and $x = 3$, but this has no effect on the receding horizon controller, which is determined bu $u(0)$.