Examination
SSY130 Applied Signal Processing
Suggested Solutions

14:00-18:00, January 16, 2019

Instructions

- *Responsible teacher:* Tomas McKelvey, phone number 8061. Teacher will visit the site of examination at approximately 14:45 and 16:30.
- Score from the written examination will together with course score determine the final grade according to the Course PM.
- Your preliminary grade is reported to you via email.
- Exam grading review will be held at 12:00-12:50 on January 31 in room 7430 (Landahlsrummet).

Allowed aids at exam:

- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta or copied sections from it), Formulaires et tables Mathématiques, Physique, Chimie, or similar.
- Any calculator
- One A4 size single sheet of paper with *handwritten* notes on both sides.

Note:

- The exam consists of 5 numbered problems.
- The ordering of the problems is arbitrary.
- All solutions should be well motivated and clearly presented in order to render a full score unless explicit instructions say otherwise. Unclear presentation or adding, for the problem in question, irrelevant information render a reduction of the score.
- Write solutions to each problem on a *separate* sheet of paper.
- The maximum score is 52 points.

Problems

1. Consider a complex exponential signal $x(n) = e^{j2\pi f_0 n/f_s}$ with frequency $f_0 = 1$ kHz and sampling frequency $f_s = 4$ kHz. Assume N samples $n = 0, 1, \dots, N-1$ are available for analysis.

   (a) Show that the magnitude of the DTFT calculated from the N available samples is

$$|\hat{X}(\omega)| = \left| \frac{\sin(\frac{N(\omega - 2\pi f_0)}{2f_s})}{\sin(\frac{(\omega - 2\pi f_0)}{2f_s})} \right|$$

(5pt)

   (b) Sketch the magnitude of $\hat{X}(\omega)$ given in (a) from frequency 0 to frequency $f_s = 4$ kHz for the case when $N = 8$. Make sure that the frequency locations of the maximum magnitude and the zero magnitudes are clearly marked. Use a frequency scale in kHz. (5pt)

Solution:

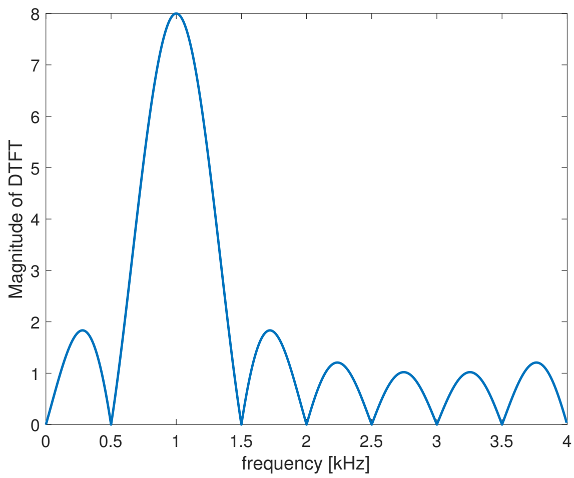
(a) Start by letting $\omega_0 = 2\pi f_0$ and $\Delta t = 1/f_s$. Then $x(n) = e^{j\omega_0 n \Delta t}$. For the DTFT we get

$$\hat{X}(\omega) = \sum_{n=0}^{N-1} e^{j\omega_0 n \Delta t}\ e^{-j\omega \Delta t n} = \sum_{n=0}^{N-1} e^{-j(\omega - \omega_0)\Delta t n}$$

$$= \frac{1 - e^{-j(\omega-\omega_0)\Delta t N}}{1 - e^{-j(\omega-\omega_0)\Delta t}} = e^{-j\frac{N-1}{2}(\omega-\omega_0)\Delta t} \frac{\sin(\frac{N(\omega-\omega_0)\Delta t}{2})}{\sin(\frac{(\omega-\omega_0)\Delta t}{2})}$$

By taking the magnitude and substituting $\Delta t = 1/f_s$ yields the desired result. We could also obtain the result by convolving the DTFT of infinte long complex exponential with the DTFT of the rectangular window function of lenght 8.

(b) The graph of the DTFT in a) for the case $N = 8$ will have a maximum of 8 for frequency 1 kHz. The sin-function in the numerator will be zero for $f = k\frac{4}{8}$ kHz for $k = 0, 1, 3, 4, 5, 6, 7$. Between these zeros we will have side lobes. The function is periodic with period 4 kHz.



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2. Consider a real valued symmetric FIR low-pass filter with a non-causal impulse response, i.e. $h_{LP}(n) = h_{LP}(-n)$. The frequency function of the filter is approximately equal to 1 in the pass-band and close to zero in the stop-band. The passband edge frequency is $0.1 f_s$.

Two bright Applied signal processing students are discussing ways to transform the low-pass filter to a high-pass filter. One student suggests to define impulse response of the high-pass filter as

$$h_{HP_1}(n) \triangleq \begin{cases} -h_{LP}(n) & n \neq 0 \\ 1 - h_{LP}(0) & n = 0 \end{cases}$$

while the other student suggests

$$h_{HP_2}(n) \triangleq (-1)^n h_{LP}(n).$$

Both students are correct.

(a) Show that $h_{HP_1}(n)$ is a high-pass filter. (3pt)

(b) Derive the passband edge frequency for $h_{HP_1}(n)$. (2pt)

(c) Show that $h_{HP_2}(n)$ is a high-pass filter. (3pt)

(d) Derive the passband edge frequency for $h_{HP_2}(n)$. (2pt)

Hint for (c-d): $(-1)^n = e^{j\pi n}$.

Solution:

(a) The frequency function of a filter is the DTFT of the impulse response. Define $h_{AP}(n) = 1$ for $n = 0$ and zero otherwise. Then

$$H_{HP_1}(\omega) = DTFT[h_{HP_1}(n)] = DTFT[h_{AP}(n) - h_{LP}(n)] = 1 - H_{LP}(\omega)$$

This shows that $H_{HP_1}(\omega)$ is close to zero where $H_{LP}(\omega)$ is close to one and $H_{HP_1}(\omega)$ is close to zero where $H_{LP}(\omega)$ is close to zero. This imply that $H_{HP_1}(\omega)$ is a high-pass filter.

(b) Since the transition region for the high-pass filter coincide with the transition region for the low-pass filter the pass-band edge frequency for the high-pass filter is also $0.1f_s$.

(c) Clearly

$$h_{HP_2}(n) = (-1)^n h_{LP}(n) = e^{j\pi n} h_{LP}(n).$$

The filter $h_{HP_2}(n)$ is hence the result from modulation with relative frequency π radians per sample which is the Nyquist frequency. The frequency response of the high-pass filter is hence

$$H_{HP_2}(\omega) = H_{LP}(\omega - \omega_s/2)$$

which clearly is a high-pass filter.

(d) The negative edge frequency at $f = -0.1f_s$ for the low pass filter is by the modulation shifted by $f_s/2$ so the edge frequency of the high-pass filter $H_{HP_2}(\omega)$ is $f_s/2 - 0.1f_s = 0.4f_s$.

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3. (a) Consider the signal $y(n)$ defined as

$$y(n) = x_0(n) + x_1(n)$$

where $x_0(n)$ and $x_1(n)$ are two uncorrelated zero mean stochastic processes with autocorrelation functions $\phi_{x_0}(n)$ and $\phi_{x_1}(n)$ respectively. Derive the autocorrelation function for $y(n)$. (4pt)

(b) Generalize the result in (a) by deriving the autocorrelation function for

$$y(n) = \sum_{k=0}^{m-1} \alpha_k x_k(n)$$

where $x_i(n)$ are zero mean stochastic processes with autocorrelation functions $\phi_{x_i}(n)$. All processes $x_i(k)$ are mutually uncorrelated. (6pt)

Solution:

(a) From the defintion of autocorrelation we get

$$\begin{aligned}\phi_{yy}(k) &= \mathbf{E}[y(n)y(n+k)] = \mathbf{E}[(x_1(n)+x_2(n))(x_1(n+k)x_2(n+k)] \\ &= \mathbf{E}[(x_1(n)x_1(n+k))] + \mathbf{E}[(x_2(n)x_2(n+k))] = \phi_{x_1}(k) + \phi_{x_2}(k)\end{aligned}$$

where the third equality follows from the uncorrelated properties of $x_1(n)$ and $x_2(n)$

(b) With the same arguments we obtain

$$\begin{aligned}\phi_y y(k) &= \mathbf{E}[y(n)y(n+k)] = \mathbf{E}[(\sum_{l=0}^{m-1} \alpha_l x_l(n))(\sum_{t=0}^{m-1} \alpha_t x_t(n+k))] \\ &= \mathbf{E}[\sum_{l=0}^{m-1} \alpha_l^2 x_l(n) x_l(n+k)] = \sum_{l=0}^{m-1} \alpha_l^2 \phi_{x_l}(k)\end{aligned}$$

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4. Multiple Choice Questions. Select one option per subquestion. No motivation is needed.

(a) A DSP system consists of a sampling unit which samples at a rate of 10 kHz, a processing unit which filters the signal using an IIR filter and a digital to analog converter (DAC) which operates in a zero-order hold (ZOH) mode. The input to the system is a single sinusoidal signal with a frequency of 4 kHz. Which statement below is *correct*: (2pt)

i. The sample rate is not high enough to ensure that no aliasing occurs during the sampling process.
ii. The output of the system is composed of infinitely many sinusoidal signals.
iii. Since no aliasing occur during the sampling the output is a single sinusoidal with the original frequency of 4 kHz.
iv. The IIR filter will only change the amplitude and not the phase of the sampled sinusoidal signal.

(b) An LMS filter can be described by the equations:

$$\begin{aligned}\hat{x}(k) &= \mathbf{h}^T \mathbf{y}(k) \\ e(k) &= x(k) - \hat{x}(k) \\ \mathbf{h} &= \mathbf{h} + 2\mu\, \mathbf{y}(k) e(k)\end{aligned}$$

where $\mathbf{h}$ is a M-length vector with the FIR filter coefficients and $\mathbf{y}(k)$ is a vector with signal samples $y(k)$ and $M-1$ past signal samples. Which statement regarding the step-length μ below is *correct*? (2pt)

i. If the filter converges, a large step-length gives a faster convergence.
ii. For stability reasons the step-length must be large enough.
iii. It is reasonable to determine the step-length based on the variance of the desired signal $x(n)$.
iv. A large step-length will give a small residual error variance.

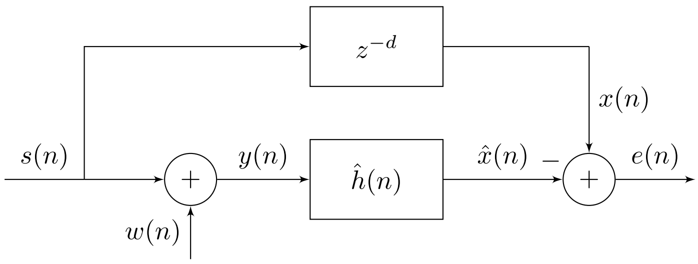
(c) The Least-Mean-Square (LMS) algorithm and the Recursive Least-Squares (RLS) algorithm can both be used for adaptive filtering. Which statement below is *incorrect:* (2pt)

i. The RLS algorithm has a more advanced step length adjustment and generally converges faster than LMS.
ii. RLS automatically selects the optimal step length.
iii. LMS is computationally more complex than the RLS algorithm.
iv. RLS with a forgetting factor is suitable when the system is slowly time-varying.

(d) When deriving the Kalman filter random variables with a Normal distribution plays a key role. Which statement below is *incorrect:* (3pt)

i. If we multiply a normally distributed random variable with a non-zero constant the result is a random variable with a normal distribution.
ii. If we add two normally distributed random variables which are un-correlated the result is a random variable with a normal distribution.
iii. The variance of the sum of two correlated random variables is the sum of the variances of the individual variables.
iv. The variance of the sum of two uncorrelated random variables is the sum of the variances of the individual variables.

(e) Consider the following block diagram:



The diagram describes a number of optimal filtering problems depending on the setting of d, the delay. The optimal filter $\hat{h}(n)$ is defined as the filter which minimizes the variance of $e(n)$. Which statement below is *incorrect*: (3pt)

i. If $d = 0$ the block diagram describes the filtering case when we want to optimally recover the signal $s(n)$ based on samples of $y(n)$ up to time index n.

ii. if $d < 0$ the block diagram describes the smoothing case, i.e. the case when we want to recover $y(n-d)$ based on samples of $y(n)$ up to time index n.

iii. If $d > 0$ the block diagram describes the smoothing case, i.e. case when we want to recover $s(n-d)$ based on samples of $y(n)$ up to time index n.

iv. if $d < 0$ the block diagram describes the prediction case, i.e. the case when we want to predict $s(n+d)$ based on samples of $y(n)$ up to time index n.

Solution:

(a) ii
(b) i
(c) iii ■
(d) iii
(e) ii

5. Monitoring the fetal status during labour is important in modern medical practice. Information derived from the fetal ECG (electrocardiogram), i.e. the electrical signature originating from the activity of the heart of the fetus, is one important modality. The fetal ECG is picked up by placing electrodes (leads) on the abdomen of the mother and viewing the electrical activity on a graphing monitor. However, the fetal ECG is quite weak due to the distance between the fetus' heart and the surface of the abdomen and is also disturbed by the ECG of the mother. This complicates the medically relevant interpretation of the fetal ECG signal. The influence of the maternal ECG can be reduced by separately measuring the ECG of the mother by placing electrodes on her chest and use the signals from the chest leads to reduce the influence of the maternal ECG on the fetal ECG measured from the abdominal leads.

(a) Assume only a single lead maternal ECG and a single lead fetal ECG is measured. Discuss a solution and draw a block diagram illustrating how an adaptive filter can be employed to suppress the maternal ECG. (5pt)

(b) It is common that several chest leads (maternal signal) and several abdominal leads (fetus signal+ maternal signal) are available for processing. Assume the signal processing system uses two chest lead signals y_1 and y_2 to remove the disturbance from one abdominal lead signal x. Hence we have

$$e(n) = x(n) - \mathbf{y}_1^T(n)\mathbf{h}_1 - \mathbf{y}_2^T(n)\mathbf{h}_2$$

Derive expressions for the optimal filters $\mathbf{h}_1^*$ and $\mathbf{h}_2^*$ which minimize the variance of the signal $e(n)$. (5pt)

Solution:

(a) Denote the maternal ECG signal from a chest lead with $y(n)$. This signal can be regarded as a measurement of the disturbance source. Denote by $x(n)$ the signal measured by an abdominal lead. This signal will be composed of the sum of the contribution originating from the maternal ECG and the fetal ECG. If we use these two signals in the system modelling setting the impulse response of the estimated filter will be a model of the signal transfer function between the chest lead and the abdominal lead. Since we can assume that the maternal and the fetal heartbeats are independent the fetal ECG signal will not (on average) affect the convergence of the adaptive filter. The filter error $e(n)$ will upon "convergence" be the improved fetal ECG signal where the maternal ECG has been suppressed.

(b) Measuring the ECG using two chest leads can improve the possibility to enhance the fetal ECG since the electric voltage is a geometrically distributed signal and sensing it at different locations will yield other wave forms.

The optimal filter minimize the variance of the error $e(n)$. At the minimum variance point the gradient w.r.t. all the filter coefficients are zero. The gradients are given by

$$\frac{d}{d\mathbf{h_1}}\,\mathbf{E}(x(n)-\mathbf{y}_1^T(n)\mathbf{h}_1-\mathbf{y}_2^T(n)\mathbf{h}_2)^2 = -2\,\mathbf{E}\,\mathbf{y}_1(n)(x(n)-\mathbf{y}_1^T(n)\mathbf{h}_1-\mathbf{y}_2^T(n)\mathbf{h}_2)$$
$$\frac{d}{d\mathbf{h_2}}\,\mathbf{E}(x(n)-\mathbf{y}_1^T(n)\mathbf{h}_1-\mathbf{y}_2^T(n)\mathbf{h}_2)^2 = -2\,\mathbf{E}\,\mathbf{y}_2(n)(x(n)-\mathbf{y}_1^T(n)\mathbf{h}_1-\mathbf{y}_2^T(n)\mathbf{h}_2)$$

Introducing the notation

$$\mathbf{h} = \begin{bmatrix}\mathbf{h}_1\\ \mathbf{h}_2\end{bmatrix}, \quad \text{and} \quad \mathbf{y}(n) = \begin{bmatrix}\mathbf{y}_1(n)\\ \mathbf{y}_2(n)\end{bmatrix}$$

and equating the gradients to zero we obtain the familiar result

$$\mathbf{E}\{\mathbf{y(n)}\mathbf{y}^T(n)\}\mathbf{h} = \mathbf{E}\{\mathbf{y}(n)x(n)\}$$

and the optimal filter is given by

$$\mathbf{h}^* = \left(\mathbf{E}\{\mathbf{y(n)}\mathbf{y^T(n)}\}\right)^{-1}\mathbf{E}\{\mathbf{y(n)}x(n)\}.$$

If $y_1(n)$ and $y_2(n)$ are uncorrelated, ($\mathbf{E}\,\mathbf{y}_1\mathbf{y}_2 = 0$) the solution is simplified to

$$\mathbf{h_1}^* = \left(\mathbf{E}\{\mathbf{y}_1(n)\mathbf{y}_1^T(n)\}\right)^{-1}\mathbf{E}\{\mathbf{y}_1(n)x(n)\}, \quad \mathbf{h_2}^* = \left(\mathbf{E}\{\mathbf{y}_2(n)\mathbf{y}_2^T(n)\}\right)^{-1}\mathbf{E}\{\mathbf{y}_2(n)x(n)\}$$

However, in the application considered here we cannot expect the two chest leads to be uncorrelated since they both sense the maternal ECG.

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END - Good Luck!