Examination
SSY130 Applied Signal Processing
Suggested Solutions

14:00-18:00, January 11, 2017

Instructions

- *Responsible teacher:* Tomas McKelvey, ph 8061. Teacher will visit the site of examination at approximately 14:45 and 16:30.
- Score from the written examination will together with course score determine the final grade according to the Course PM.
- Solutions are published on the course home-page latest on January 14
- Your preliminary grade is reported to you via email.
- Exam grading review will be held at 12:00-13:00 on January 27 in room 7430 Landahlsrummet.

Allowed aids at exam:

- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta or copied sections from it), Formulaires et tables Mathématiques, Physique, Chimie, or similar.
- Any calculator
- One A4 size single sheet of paper with *handwritten* notes on both sides.

Note:

- The exam consists of 5 numbered problems.
- The ordering of the questions is arbitrary.
- All solutions should be well motivated and clearly presented in order to render a full score. Unclear presentation or adding, for the problem in questing, irrelevant information render a reduction of the score.
- Write solutions to each problem on a *separate* sheet of paper.
- The maximum score is 52 points.

Problems

1. Frequency selective filters are specified by the following properties

   - Stopband rejection
   - Passband ripple
   - Passband edge frequency
   - Stopband edge frequency
   - Width of transition region

   (a) Draw a frequency function of a high-pass filter and illustrate in the graph how the specifications above are defined. (4pt)

   (b) Filter design involves a fundamental tradeoff which limits the performance of a filter of a given length. Describe this tradeoff. (2pt)

   Solution:

   (a) The definitions for a LP filter are as given in Figure 1. For a HP filter simply mirror the image. Stopband rejection is in the figure denoted by stopband attenuation.

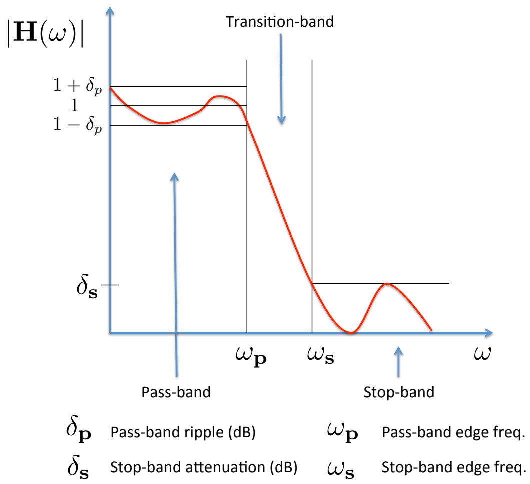

Figure 1: Illustration of filter specifications for a frequency selective filter.

   (b) For a given filter length the main trade-off is between on one hand having a narrow transition region and on the other hand have a small passband ripple and high attenuation in the stopband.

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2. In the following subproblems you will get 1 point if the answer is correct and 1 point if the answer is correctly motivated. All frequency scales are relative to the sampling frequency, i.e. 1 corresponds to the sampling frequency.

   (a) The magnitude of the DTFT of two FIR filters designed with the Parks-McClellan optimal equiripple FIR filter design method are shown in Figure 2 (top left). Which of the two filters have the longest length. (2pt)

   (b) Two high-pass filters of the same length are designed using the window method. Filter A is designed using the window function A and filter B is designed with window function B. The DTFT of the two window functions are shown in Figure 2 (top right).

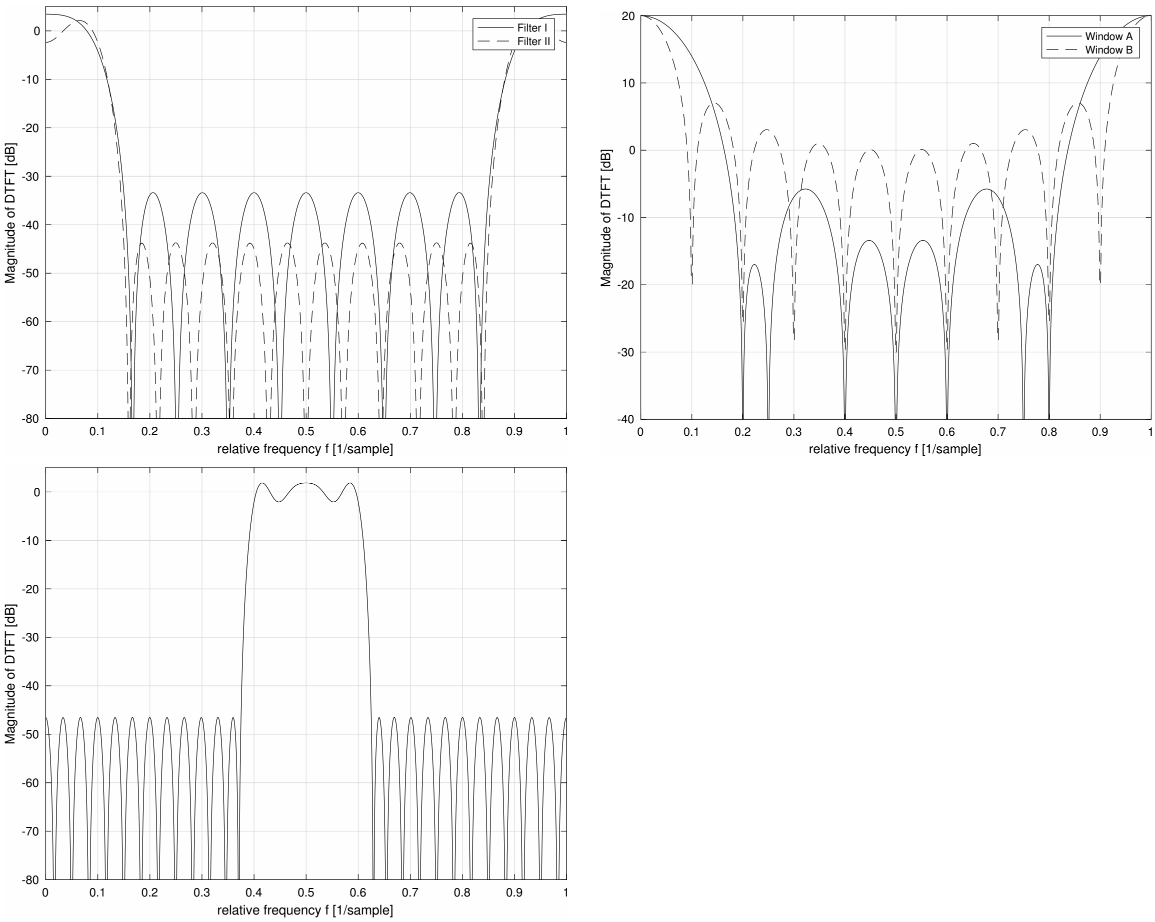

Figure 2: Magnitude DTFT of two FIR filters (top left). Magnitude DTFT of two window functions (top right). Magnitude DTFT of the impulse response of a filter (bottom left).

i. Which of the two filters will have the best stop-band attenuation? (2pt)

ii. Which of the two filters will have the largest transition band? (2pt)

(c) The magnitude of the DTFT of the impulse response of a filter is shown in Figure 2 (bottom left).

i. Describe the type of filter (HP,LP,BP or BS) (2pt)

ii. What is/are the crossover frequency/frequencies in Hertz if we assume the sample rate of the filter is 20 kHz. (2pt)

Solution:

(a) A longer filter means a more flexible filter. A flexible filter can be made to have better performance, for example better stop band attenuation. ⇒ Filter II is the longest.

(b) i. The frequency function for this design is the result of the frequency domain convolution between the ideal LP-filter and the DTFT of the window function. If the window has small side lobes ⇒ the filter will have a high stop band attenuation. Since Window A has the lowest side lobes ⇒ Filter A will have the best stop-band attenuation.

ii. Again due to the convolution the width of the transition band is proportional to the width of the main lobe of the window function. ⇒ Filter A will have the largest transition band.

(c) i. Since the DTFT is plotted in relative frequency where 1=sampling frequency and 0.5 is Nyquist frequency. Hence filter is a High-Pass filter (HP).

ii. The crossover frequency for the filter is at relative frequency 0.4 which in Hertz is $f_c = 0.4 * 20$ kHz $= 8$ kHz.

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3. It is desired to perform frequency analysis of a narrow-band real-valued signal which has a bandwidth of 10 MHz and has a lowest frequency of 1.00 GHz (and hence a highest frequency of 1.01 GHz). Instead of using a demodulator based solution (like in Project 1B) the signal is directly sampled using a sample rate which is much lower than the rate required by the Nyquist theorem, but high enough to accomodate the band-width of the signal. This is known as *bandpass-sampling*. After the downsampling the frequency analysis is performed using the DFT.

(a) Show that the minimal sample rate for the bandpass-sampling solution which preserves the information in the original bandpass-signal is 20 MHz. (5pt)

(b) When using the DFT, the frequency resolution is inversely proportional to the number time domain of samples of the signal under analysis. How many samples do we need to use in order to obtain a frequency resolution of 100 kHz in the DFT analysis. (2pt)

(c) For what duration (in seconds) must we observe the original signal in order to obtain a frequency resolution of 100 kHz in the DFT analysis. (2pt)

(d) Assume the original signal is composed of 2 sinusoidal signals with frequencies within the band (1.00-1.01 GHz) and the DFT based on 1000 samples is calculated. The indices of the four DFT coefficients with the largest magnitudes of are 350, 450, 550 and 650. What frequencies in Hz does the two original sinusoidal signals have? (4pt)

Solution:

(a) The sampled signal will have a DTFT corresponding to

$$X_d(\omega) = X_c(\omega) = \frac{1}{\Delta t} \sum_{k=-\infty}^{\infty} X(\omega + \frac{2\pi}{\Delta t} k)$$

The DTFT of the sampled signal is the sum of all 20 MHz segments of the Fourier transform of the original signal. Since the lowest frequency is aligned with the an integer multiple of the sample rate the frequency 1.0 GHz will be folded down to 0 and frequency 1.010 GHz will be folded down to 10 MHz. The signal is real valued so the negative frequency content of the signal will also fold down to 10 - 20 MHz where 20 MHz corresponds to -1.00 GHz and 10 MHz corresponds to 10 MHz. Hence the folded spectra does not overlap and all the information is preserved.

(b) Since the sample rate is 20 MHz the required number of samples are 20 MHz / 100 kHz = 20 / 0.1 = 200 samples.

(c) Since the sample rate is 20 MHz we obtain the time to be 200 / 20E6 = 0.1 μs.

(d) The real valued sinusoidal signal will produce 4 peaks in the DFT. The two with lowest indices correspond to the positive frequencies while the two with higher indices correspond to the the negative frequencies. Hence, the frequencies of the two sinusoidal are 1 GHz + fs * k/N which will be 1 GHz + 20 MHz * 350/1000 = 1.007 GHz and 1 GHz + 20 MHz * 450/1000 = 1.009 GHz

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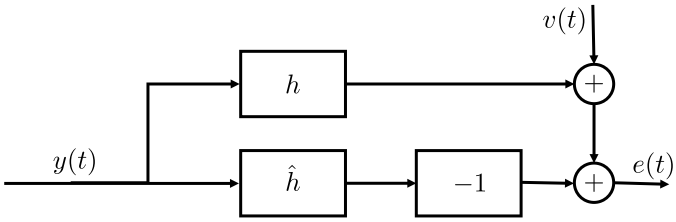

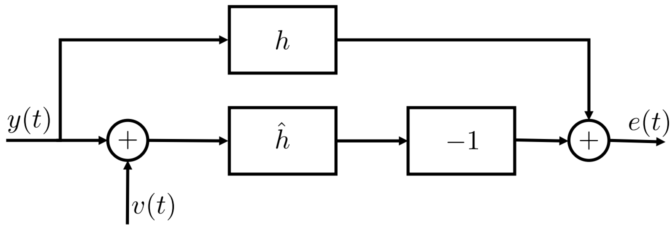

Figure 3: The two optimal filtering cases. Top graph case (a). Bottom graph case (b).

4. Consider the two optimal filtering cases in Figure 3 where the signals y and v are zero mean white random processes with covariance σ_y^2 and σ_v^2 respectively. Signals y and v are also uncorrelated. The filter h is of FIR type and has the frequency function

$$H(\omega) = h_0 + h_1 e^{-j\omega} + h_2 e^{-j2\omega}$$

(a) Determine the optimal FIR filter $\hat{h}$ of the same size as h which minimize the variance of the error signal e for both cases. (10pt)

(b) Assume we use the LMS method in order to estimate the impulse response of the filter h. Clearly only one of the cases provides the correct solution. Discuss how the estimate can be modified and what additional information (as compared to the other case) is needed to make the estimate correct. (5pt)

Solution:

(a) For case (a) we have

$$\mathbf{E}\, e^2 = \mathbf{E}\left((h_0 - \hat{h}_0)y(n) + (h_1 - \hat{h}_1)y(n-1) + (h_2 - \hat{h}_2)y(n-2) + v(n)\right)^2$$

Since y and v are zero mean and uncorrelated the minimum is obtained for $\hat{h}_i = h_i$, $i = 0, 1, 2$. (4pt)

For case (b) we obtain (again using that y and v are zero mean and uncorrelated

$$\begin{aligned}
\mathbf{E}\, e^2 &= \mathbf{E}\left(\sum_{k=0}^{2}(h_k - \hat{h}_k)y(n-k) - \sum_{k=0}^{2}\hat{h}_k v(n-k)\right)^2 \\
&= \mathbf{E}\left(\sum_{k=0}^{2}(h_k - \hat{h}_k)y(n-k)\right)^2 + \left(\sum_{k=0}^{2}\hat{h}_k v(n-k)\right)^2 \\
&= \sum_{k=0}^{2}(h_k - \hat{h}_k)^2\sigma_y^2 + \sum_{k=0}^{2}\hat{h}_k^2\sigma_v^2
\end{aligned}$$

Since the variance is bounded from below by zero and the problem is quadratic an a minimum value exists. At this point the gradient is zero. Taking the gradient w.r.t. $\hat{h}_i$

and solve for the zero solution yields

$$-2(h_i - \hat{h}_i)\sigma_y^2 + 2\hat{h}_i\sigma_v^2$$

Hence for case (b) the optimal filter, for $k = 0, 1, 2$, is given by

$$\hat{h}_i = h_i \frac{\sigma_y^2}{\sigma_v^2 + \sigma_y^2} = h_i \frac{1}{1 + \sigma_v^2/\sigma_y^2}$$

(6pt)

(b) Using the LMS algorithm the estimated filter coefficients will converge towards the optimal solution (i.e. when the variance of e is minimized). Clearly for case (b) the LMS filter will *not* converge to the FIR filter h. However if we, a-priori,m know the ratio σ_v^2/σ_y^2 we can, a-posteri, compensate the estimate obtained from the LMS algorithm by scaling:

$$\hat{\hat{h}}_i = \hat{h}_i(1 + \sigma_v^2/\sigma_y^2)$$

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5. This problem concerns using the DFT/FFT to perform the convolution operation. Assume $x(n)$ is a signal and $h(n)$ is a signal (e.g. the impulse response of a filter) and we want to derive the output $y(n)$ from convolution

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

for all values of n. Explain how the radix-2 FFT can be used to achieve this for the following situations

(a) when $x(n)$ is a signal of finite length N and $h(n)$ is a signal of finite length M. (4p)

(b) when $x(n)$ is a periodic signal with period $P = 2^L$ for some positive integer L and $h(n)$ is a signal of finite length $M \leq P$. (4p)

Solution:

(a) The linear convolution will yield an output $y(n)$ which is zero for $n < 0$ and $n \geq N_y = N + M - 1$. The output is of length N_y and is zero otherwise. From DTFT theory we know that convolution is equivalent with $Y(\omega) = H(\omega)X(\omega)$ for all values of ω. Recall that DFT and DTFT are related for finite signals as $Y(\omega_k) = Y(k)$ where $Y(k)$ is the DFT of length N_y and $\omega_k = 2\pi k/N_h$. We also know that given $Y(k)$ the inverse DFT will yield $y(n)$. Hence we need to calculate the product of $H(\omega)$ and $X(\omega)$ at the frequencies $\omega_k = \frac{2\pi k}{N_y}$ for $k = 0, \ldots, N_y - 1$. Since both N and M are less than N_y and both signals are zero outside their respective interval we can zero pad both signals to obtain the length N_y and then calculate the DFT to obtain the desired product. In MATLAB this would look like

```
>> N = length(x);
>> M = length(h);
>> Ny = N+M-1;
>> X = fft(x,Ny);
>> H = fft(h,Ny);
>> Y = H.*X;
>> y = ifft(Y);
```

To make this execute fast with the Radix-2 FFT algorithm, N_y should be selected to be the nearest power of 2 that is larger or equal to $N + M - 1$.

(b) Since the input is periodic the output will also be a periodic signal with period P. From signals and systems theory we can thus express this signal using the Fourier series approach and we have:

$$x(n) = \frac{1}{P}\sum_{k=0}^{P-1} X(k)e^{j\frac{2\pi kn}{P}}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{1}$$

where $X(k)$ is the DFT of one period of the input $x(n)$, i.e. for $n = 0, \ldots, P-1$. Filtering this signal through the filter with frequency function $H(\omega)$ consequently yields the output

$$y(n) = \frac{1}{P}\sum_{k=0}^{P-1} H(\omega_k)X(k)e^{j\frac{2\pi kn}{P}}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{2}$$

where $\omega_k = 2\pi k/P$. The complex values $H(\omega_k)$ is obtained by calculating the DFT of the impulse response zero-padded to a length P. In MATLAB this can be formulated as (for the case when $M \leq P$:

```
>> P = length(x); % X assumed P-periodic
>> X = fft(x,P);
>> H = fft(h,P);
>> Y = X.*H; % Circular convolution
>> y = ifft(Y);
```

Since, by assumption, $P = 2^L$ and L integer, the Radix-2 FFT algorithm can be used.

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END - Good Luck!