# Suggested Solutions to Examination SSY130 Applied Signal Processing

14:00-18:00, December 20, 2012

**Problems**

1. A sampled signal is known to contain 3 real valued sinusoidal signals. Two of the signals have equal amplitude and are large while the third signal has an amplitude which is 5 times smaller. The signal is also contaminated with white noise. Figure 1 shows the magnitude of the DFT of the zero padded sampled signal with a total length of 100 samples. The locations of the 5 largest peaks are indicated in the graph. The location is given as the value of the DFT index $k$.

   (a) Assume all sinusoidal signals have a frequency less than half the sampling frequency. Give an estimate of the relative frequencies (i.e. relative to the sampling frequency) of the 3 sinusoidal signals. (4pt)

   (b) Explain why the magnitude of the DFT have more than 3 peaks? (2pt)

   **Solution:**

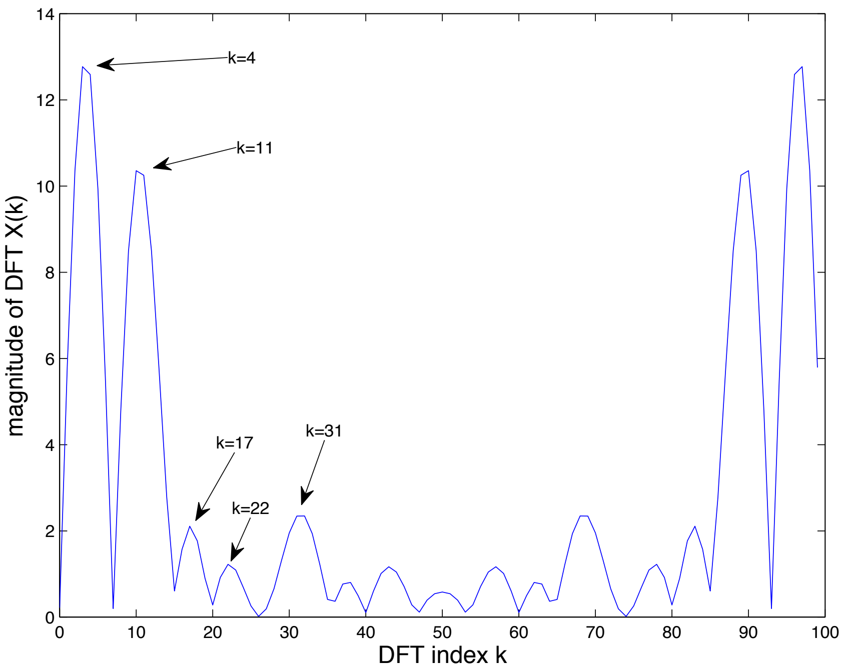


Figure 1: Magnitude of DFT.

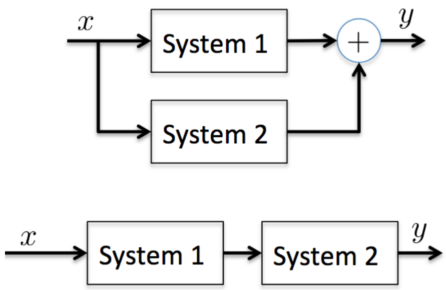


Figure 2: Block diagram of parallell and serial filter connections.

(a) Since all signals are real valued and have frequencies below the Nyquist frequency the peak locations below relative frequency 0.5 will give a good estimate of the freqency. Since two signals are larger than the third we expect to see two dominant peaks and one peak which is 5 times lower then the dominant ones. Peaks at locations $k = 4$, 11 and 31 är then suitable candidates. The relative frequencies are given as the index dividet with the number of signal samples. Estimated frequencies are then 4/100=0.04, 11/100=0.11 and 31/100=0.31.

(b) Since the signal is real valued the DFT will also have three peaks corresponding to the negative frequencies. In the graph these are located above the Nyquist frequencies. The other peaks are the results from the sidelobes of the rectangular window and the noise. Clearly if the level of the third sinusoid would be even lower it would be difficult to make a correct estimation from the graph.

2. Consider the block diagrams in Figure 2. The blocks marked System 1 and Systm 2 are FIR filters. System 1 and System 2 have the impulse responses

$$h_1(k) = \begin{cases} 1, & k = 0 \\ 2, & k = 1 \\ 3, & k = 2 \\ 4, & k = 3 \\ 0, & \text{otherwise} \end{cases} \qquad h_2(k) = \begin{cases} 1, & k = 0 \\ 2, & k = 1 \\ 4, & k = 2 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Derive the impulse response of the equivalent filter for the upper blockdiagram in Figure 2, i.e. the filter which satisfy $y(n) = h(n) * x(n)$ (3pt)

(b) Derive the impulse response of the equivalent filter for the lower filter in Figure 2, i.e. the filter which satisfy $y(n) = h(n) * x(n)$ (3pt)

(c) Derive the impulse response of the equivalent filter for the lower filter in Figure 2, i.e. the filter which satisfy $y(n) = h(n) * x(n)$ when the blocks System 1 and System 2 have changed places. (3pt)

**Solution:**

(a) For the upper filter we obtain $H(z) = \sum_{k=0}^{3} h_1(k)z^{-k} + \sum_{k=0}^{2} h_2(k)z^{-k} = \sum_{k=0}^{3}(h_1(k) + h_2(k))z^{-k}$, where we set $h_2(3) = 0$ this yields the impulse response $\{1 + 1, 2 + 2, 3 + 4, 4 + 0\} = \{2, 5, 7, 4\}$.

(b) The equivalent filter for the lower diagram is a filter with an impulse response which is the convolution of the impulse responses of filter $H_1$ and $H_2$ respectively.

$$H(z) = \sum_{k=0}^{4+3-2} (\sum_{n=0}^{3} h_1(n)h_2(k-n))z^{-k}$$

and we obtain

$$\begin{aligned}
h(0) &= 1 \times 1 = 1 \\
h(1) &= 1 \times 2 + 2 \times 1 = 4 \\
h(2) &= 1 \times 4 + 2 \times 2 + 3 \times 1 = 11 \\
h(3) &= 2 \times 4 + 3 \times 2 + 4 \times 1 = 18 \\
h(4) &= 3 \times 4 + 4 \times 2 = 20 \\
h(5) &= 4 \times 4 = 16
\end{aligned}$$

(c) Since convolution is commutative (i.e. the order of the operations does not change the result) the result is identical as in (2b).

3. An audio signal is received by a microphone, sampled and stored by a DSP system. The signal has bandpass character and contain power only between 8 kHz and 8.4 kHz. Suggest a bandpass sampling solution which

   - prevents aliasing
   - use as low sampling frequency as possible to minimize the memory requirements for signal storage. Your answer should include the sample frequency of the processing system. (7pt)

**Solution:** The bandwidth of 400 Hz suggest a sample rate of minimum 800 Hz if we do not get aliasing problems due to the position of the frequency band. We now check that 800 Hz will not lead to aliasing. Sampling of the raw signal with FT $X_c(\omega)$ will yield a sampled signal with the DTFT

$$X(2\pi f) = f_s \sum_{k=-\infty}^{\infty} X_c(2\pi f + k2\pi f_s) \tag{1}$$

$X_c(2\pi f)$ is non-zero for the segments $f = [-8.4, -8]$ kHz and $[8, 8.4]$ kHz. For $f = 0$ to 400 Hz only one term in (1) ($k = 10$) is non-zero and we obtain $X(2\pi f) = 800 X_c(2\pi f + 10 \times 2\pi 800)$. For $f = 400$ to 800 Hz the only non-zero term in (1) is the corresponding negative part ($k = -11$) and we obtain $X(2\pi f) = 800 X_c(2\pi f - 11 \times 2\pi 800)$. Hence, we have established that aliasing does not occur for the sample frequency 800 Hz.

4. Monitoring the fetal status during labour is important in modern medical practice. Information derived from the fetal ECG (electrocardiogram), i.e. the electrical signature originating from the activity of the heart of the fetus, is one important modality. The fetal ECG is picked up by placing electrodes (leads) on the abdomen of the mother and viewing the electrical activity on a graphing monitor. However, the fetal ECG is quite weak and is also seriously disturbed by the ECG of the mother. This complicates the medically relevant interpretation of the fetal ECG signal. The influence of the maternal ECG can be reduced by separately measure the ECG of the mother by placing electrodes on her chest and use the signals from the chest leads to reduce the influence of the maternal ECG on the fetal ECG measured from the abdominal leads.

   (a) Motivate why an adaptive filter solution as compared to a fixed filter solution could be useful to solve the maternal ECG suppression problem. (3pt)

   (b) Assume only a single lead maternal ECG and a single lead fetal ECG is measured. Discuss a solution and draw a block diagram illustrating how an adaptive filter can be employed to suppress the maternal ECG. (5pt)

   (c) It is common that several chest leads (maternal signal) and several abdominal leads (fetus signal+ maternal signal) are available for processing. Assume the signal processing system uses two chest lead signals $y_1$ and $y_2$ to remove the disturbance from an abdominal lead signal $x$. Hence we have

$$e(n) = x(n) - \mathbf{y}_1^T(n)\mathbf{h}_1 - \mathbf{y}_2^T(n)\mathbf{h}_2$$

Derive expressions for the optimal filters $\mathbf{h}_1^*$ and $\mathbf{h}_2^*$ which minimize the variance of the signal $e(n)$. (5pt)

(d) Give example of a pseudo-program which implement an LMS adaptive filter solution based on the results in problem (4c) (4pt)

**Solution:**

(a) Clearly the maternal ECG is much stronger than the fetal ECG so we can assume the chest ECG leads will only pick upp the maternal ECG. At the abdominal leads we measure the sum of the chest EEG filtered through some unknown filter and the fetal ECG. The properties of the filter will most likely vary with the induvidual as well as the exact location of the leads on the body. Hence in order to subtract the maternal ECG from the fetal ECG the filter must be estimated for each measurment situation and also adaptively follow any changes in the signal transmission path. An adaptive solution is hence a viable solution.

(b) Denote the maternal ECG signal from a chest lead with $y(n)$. This signal can be regarded as a measurement of the disturbance source. Denote by $x(n)$ the signal measured by an abdominal lead. This signal will be composed of the sum of the contribution originating from the maternal ECG and the fetal ECG. If we use these two signals in the system modelling setting the impulse response of the estimated filter will be a model of the signal transfer function between the chest lead and the abdominal lead. Since we can assume that the maternal and the fetal heartbeats are independent the fetal ECG signal will not (on average) affect the convergence of the adaptive filter. The filter error $e(n)$ will upon "convergence" be the improved fetal ECG signal where the maternal ECG has been suppressed.

(c) Measuring the ECG using two chest leads can improve the possibility to enhance the fetal ECG since the electric voltage is a geometrically distributed signal and sensing it at different locations will yield other wave forms.

The optimal filter minimize the variance of the error $e(n)$. At the minimum variance point the gradient w.r.t. all the filter coefficients are zero. The gradients are given by

$$\frac{d}{d\mathbf{h_1}}\,\mathbf{E}(x(n) - \mathbf{y}_1^T(n)\mathbf{h}_1 - \mathbf{y}_2^T(n)\mathbf{h}_2)^2 = -2\,\mathbf{E}\,\mathbf{y}_1(n)(x(n) - \mathbf{y}_1^T(n)\mathbf{h}_1 - \mathbf{y}_2^T(n)\mathbf{h}_2)$$
$$\frac{d}{d\mathbf{h_2}}\,\mathbf{E}(x(n) - \mathbf{y}_1^T(n)\mathbf{h}_1 - \mathbf{y}_2^T(n)\mathbf{h}_2)^2 = -2\,\mathbf{E}\,\mathbf{y}_2(n)(x(n) - \mathbf{y}_1^T(n)\mathbf{h}_1 - \mathbf{y}_2^T(n)\mathbf{h}_2)$$

Introducing the notation

$$\mathbf{h} = \begin{bmatrix}\mathbf{h}_1\\ \mathbf{h}_2\end{bmatrix}, \quad \text{and} \quad \mathbf{y}(n) = \begin{bmatrix}\mathbf{y}_1(n)\\ \mathbf{y}_2(n)\end{bmatrix}$$

and equating the gradients to zero we obtain the familiar result

$$\mathbf{E}\{\mathbf{y}(\mathbf{n})\mathbf{y}^T(n)\}\mathbf{h} = \mathbf{E}\{\mathbf{y}(n)x(n)\}$$

and the optimal filter is given by

$$\mathbf{h}^* = \left(\mathbf{E}\{\mathbf{y}(\mathbf{n})\mathbf{y}^\mathbf{T}(\mathbf{n})\}\right)^{-1}\mathbf{E}\{\mathbf{y}(\mathbf{n})x(n)\}.$$

If $y_1(n)$ and $y_2(n)$ are uncorrelated, ($\mathbf{E}\,\mathbf{y}_1\mathbf{y}_2 = 0$) the solution is simplified to

$$\mathbf{h_1}^* = \left(\mathbf{E}\{\mathbf{y}_1(n)\mathbf{y}_1^T(n)\}\right)^{-1}\mathbf{E}\{\mathbf{y}_1(n)x(n)\}, \quad \mathbf{h_2}^* = \left(\mathbf{E}\{\mathbf{y}_2(n)\mathbf{y}_2^T(n)\}\right)^{-1}\mathbf{E}\{\mathbf{y}_2(n)x(n)\}$$

However, in the application considered here we cannot expect the two chest leads to be uncorrelated since they both sense the maternal ECG.

(d) Since the optimal solution has the same structure for two channels of the signal $y(n)$ the LMS algorithm can be extended in a straight forward fashion where we approximate the negative gradient w.r.t. $\mathbf{h}_i$ with the product $\mathbf{y}_i(n)e(n)$. After calculating the error $e(n)$, each of the filters are updated according to the approximate negative gradient direction. A MATLAB code can be formulated as

```
for k=Nh:1000-Nh,
  e(k) = x(k) - h1'*y1(k-Nh+1:k) - h2'*y2(k-Nh+1:k) ;
  h1 = h1 + mu*y1(k-Nh+1:k)*e(k);
  h2 = h2 + mu*y2(k-Nh+1:k)*e(k);
end
```

where `Nh` is the length of the two FIR filters.

5. In multi-rate processing decimation and interpolation functions plays a key role. Consider an interpolation stage with a rate change factor of $L$ which can be factorized into $I$ integer factors , i.e., $L = L_1 L_2 \dots L_I$. Discuss possible advantages and disadvantages with implementing the interpolation function as a cascaded version of $I$ interpolation stages including $I$ filters and $I$ up-sampling units. Things to consider are for example filter lengths, necessary storage for filter coefficients, computational complexity (multiplications per second), etc. (5pt)

**Solution:** Consider first a single stage interpolator design with an factor $L$ up-sampling unit followed by a low pass filter with cut off frequency at $\frac{f_s}{2L}$. Assume the required filter length is $N$ to meet transition region and stop-band attenuation specifications. If the interpolator instead is designed using a cascade of two interpolators the LP-filtering requirements will be divided between the LP-filters in the two stages. Assume the filter lengths are $N_1$ and $N_2$ in the cascaded stages respectively. The number of multiplications needed per output sample of the complete interpolator is $N$ for the one stage interpolator and $N_1/2 + N_2$ for the two stage interpolator. We can argue that if $N_1 + N_2 = N$ we have the possibility to obtain this. Further assume $N_1 = N_2$ to balance the computations. Then the number of multiplications per output sample will be $3N/4$ for the cascaded solution compared to $N$ for the single stage solution. A reduction of 25% is thus achieved.

Furthermore, the adverse effects of using finite word lengths are less pronounced when using a short filter as compared to a long filter which also is in favor of a cascaded solution.

Finally, if $L_i = L_j$ for all $i, j$, it is possible to use the same filter coefficients (i.e. the same LP-filter) in all stages. Hence, only one (shorter) impulse response is required to be stored in memory.

6. This problem concern using DFT/FFT to calculate the result of convolution. Assume $x(n)$ is a signal and $h(n)$ is a signal (e.g. the impulse response of a filter) and we want to derive the output $y(n)$ from convolution

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

for all values of $n$. Explain how the radix-2 FFT can be used to achieve this for the following situations

(a) when $x(n)$ is a signal of finite length $N$ and $h(n)$ is a signal of finite length $M$. (4p)

(b) when $x(n)$ is a periodic signal with period $P = 2^L$ for some positive integer $L$ and $h(n)$ is a signal of finite length $M \leq P$. (4p)

**Solution**

(a) The linear convolution will yield an output $y(n)$ which is zero for $n < 0$ and $n \geq N_y = N + M - 1$. The output is of length $N_y$ and is zero otherwise. From DTFT theory we now that convolution is equivalent with $Y(\omega) = H(\omega)X(\omega)$ for all values of $\omega$. Recall that DFT and DTFT are related for finite signals as $Y(\omega_k) = Y(k)$ where $Y(k)$ is the DFT of length $N_y$ and $\omega_k = 2\pi k/N_h$. We also know that given $Y(k)$ the inverse DFT will yield $y(n)$. Hence we need to calculate the product of $H(\omega)$ and $X(\omega)$ at the frequencies $\omega_k = \frac{2\pi k}{N_y}$ for $k = 0, \ldots, N_y - 1$. Since both $N$ and $M$ are less than $N_y$ and both signals are zero outside their respective interval we can zero pad both signals to obtain the length $N_y$ and then calculate the DFT to obtain the desired product. In MATLAB this would look like

```
>> N = length(x);
>> M = length(h);
>> Ny = N+M-1;
>> X = fft(x,Ny);
>> H = fft(h,Ny);
>> Y = H.*X;
>> y = ifft(Y);
```

To make this execute fast with the Radix-2 FFT algorithm, $N_y$ should be selected to be the nearest power of 2 that is larger or equal to $N + M - 1$.

(b) Since the input is periodic the output will also be a periodic signal with period $P$. From signals and systems theory we can thus express this signal using the Fourier series approach and we have:

$$x(n) = \frac{1}{P}\sum_{k=0}^{P-1} X(k)e^{j\frac{2\pi kn}{P}}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{2}$$

where $X(k)$ is the DFT of one period of the input $x(n)$, i.e. for $n = 0, \ldots, P - 1$. Filtering this signal through the filter with frequency function $H(\omega)$ consequently yields the output

$$y(n) = \frac{1}{P}\sum_{k=0}^{P-1} H(\omega_k)X(k)e^{j\frac{2\pi kn}{P}}, \quad n = 0, \pm 1, \pm 2, \ldots \tag{3}$$

where $\omega_k = 2\pi k/P$. The complex values $H(\omega_k)$ is obtained by calculating the DFT of the impulse response zero-padded to a length $P$. In MATLAB this can be formulated as (for the case when $M \leq P$:

```
>> P = length(x); % X assumed P-periodic
>> X = fft(x,P);
>> H = fft(h,P);
>> Y = X.*H; % Circular convolution
>> y = ifft(Y);
```

Since, by assumption, $P = 2^L$ and $L$ integer, the Radix-2 FFT algorithm can be used.

END - Good Luck!