Examination
SSY130 Applied Signal Processing

14:00-18:00, December 13, 2011

Instructions

- *Responsible teacher:* Tomas McKelvey, ph 8061. Teacher will visit the site of examination at approximately 14:45 and 16:30.
- Score from the written examination will together with course score determine the final grade according to the Course PM.
- Solutions are published on the course home-page latest on December 15.
- Your preliminary grade is reported to you via email.
- Exam grading review will be held in room 6439 "Blue Room" on level 6 at 12:30-13:00 on January 19, 2011.

Allowed aids at exam:

- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta).
- Any calculator
- One a4 size single sheet of paper with written notes on both sides.

Other important issues:

- The ordering of the questions is arbitrary.
- All solutions should be well motivated and clearly presented in order to render a full score. Unclear presentation or adding, for the problem in questing, irrelevant information render a reduction of the score.
- Write solutions to each problem on a separate sheet of paper.
- The maximum score is 52 points.

Problems

1. The optimal filter defined by the minimizing argument to

$$L_N(\mathbf{h}) \triangleq \frac{1}{N+1}\sum_{n=0}^{N} e^2(n)\alpha^{N-n} = \frac{1}{N+1}\sum_{n=0}^{N}\left(x(n) - \mathbf{h}^T\mathbf{y}(n)\right)^2\alpha^{N-n} \tag{1}$$

can be calculated by solving

$$\hat{\mathbf{h}}(N) = \mathbf{R}_{yy}^{-1}(N)\mathbf{R}_{yx}(N). \tag{2}$$

where

$$\mathbf{R}_{yy}(N) \triangleq \sum_{n=0}^{N}\alpha^{N-n}\mathbf{y}(n)\mathbf{y}^T(n), \quad \mathbf{R}_{yx}(N) \triangleq \sum_{n=0}^{N}\alpha^{N-n}\mathbf{y}(n)x(n). \tag{3}$$

Show that this solution can be derived recursively by the RLS algorithm given by

$$\hat{\mathbf{h}}(N) = \hat{\mathbf{h}}(N-1) + \mathbf{R}_{yy}^{-1}(N)\mathbf{y}(N)e(N) \tag{4}$$

$$\mathbf{R}_{yy}^{-1}(N) = \frac{1}{\alpha}\left[\mathbf{R}_{yy}^{-1}(N-1) - \frac{\mathbf{R}_{yy}^{-1}(N-1)\mathbf{y}(N)\mathbf{y}^T(N)\mathbf{R}_{yy}^{-1}(N-1)}{\alpha + \mathbf{y}^T(N)\mathbf{R}_{yy}^{-1}(N-1)\mathbf{y}(N)}\right] \tag{5}$$

Hint: Consider the symmetric matrix A defined as

$$A = B + vv^T \tag{6}$$

where v is a column vector and B is a symmetric matrix. If B^{-1} exists then the inverse of A is given as

$$A^{-1} = B^{-1} - \frac{B^{-1}vv^TB^{-1}}{1 + v^TB^{-1}v} \tag{7}$$

(10pt)

Solution: Clearly from the definition in (3) we have

$$\mathbf{R}_{yy}(N) = \alpha\mathbf{R}_{yy}(N-1) + \mathbf{y}(N)\mathbf{y}^T(N) \tag{8}$$

and

$$\mathbf{R}_{yx}(N) = \alpha\mathbf{R}_{yx}(N-1) + \mathbf{y}(N)x(N) \tag{9}$$

Using relation (2) on the left hand side in (9) and relation (8) on the right hand side we obtain

$$\begin{aligned}\mathbf{R}_{yy}(N)\hat{\mathbf{h}}(N) &= \alpha\mathbf{R}_{yy}(N-1)\hat{\mathbf{h}}(N-1) + \mathbf{y}(N)x(N)\\ &= \left(\mathbf{R}_{yy}(N) - \mathbf{y}(N)\mathbf{y}^T(N)\right)\hat{\mathbf{h}}(N-1) + \mathbf{y}(N)x(N)\end{aligned} \tag{10}$$

Furthermore expanding the parenthesis yields

$$\begin{aligned}&\mathbf{R}_{yy}(N)\hat{\mathbf{h}}(N)\\ &\quad= \mathbf{R}_{yy}(N)\hat{\mathbf{h}}(N-1) - \mathbf{y}(N)\underbrace{\mathbf{y}^T(N)\hat{\mathbf{h}}(N-1)}_{\hat{x}(N)} + \mathbf{y}(N)x(N)\\ &\quad= \mathbf{R}_{yy}(N)\hat{\mathbf{h}}(N-1) + \mathbf{y}(N)\underbrace{\left(x(N) - \hat{x}(N)\right)}_{e(n)}\\ &\quad= \mathbf{R}_{yy}(N)\hat{\mathbf{h}}(N-1) + \mathbf{y}(N)e(N)\end{aligned} \tag{11}$$

Assuming $\mathbf{R}_{yy}(N)$ non-singular we obtain the recursion for the filter as

$$\hat{\mathbf{h}}(N) = \hat{\mathbf{h}}(N-1) + \mathbf{R}_{yy}^{-1}(N)\mathbf{y}(N)e(N) \tag{12}$$

which is the desired result in equation (4). Since the formation of $\mathbf{R}_{yy}(N)$ in (8) is a rank one update, using the result in (6) and (7) with $B = \frac{1}{\alpha}\mathbf{R}_{yy}^{-1}(N-1)$ and $v = \mathbf{y}(N)$ the inverse of $\mathbf{R}_{yy}(N)$ can be formulated as a recursion:

$$\begin{aligned}\mathbf{R}_{yy}^{-1}(N) &= \frac{1}{\alpha}\mathbf{R}_{yy}^{-1}(N-1) - \frac{\frac{1}{\alpha^2}\mathbf{R}_{yy}^{-1}(N-1)\mathbf{y}(N)\mathbf{y}^T(N)\mathbf{R}_{yy}^{-1}(N-1)}{1+\mathbf{y}^T(N)\frac{1}{\alpha}\mathbf{R}_{yy}^{-1}(N-1)\mathbf{y}(N)} \\ &= \frac{1}{\alpha}\left[\mathbf{R}_{yy}^{-1}(N-1) - \frac{\mathbf{R}_{yy}^{-1}(N-1)\mathbf{y}(N)\mathbf{y}^T(N)\mathbf{R}_{yy}^{-1}(N-1)}{\alpha+\mathbf{y}^T(N)\mathbf{R}_{yy}^{-1}(N-1)\mathbf{y}(N)}\right]\end{aligned} \tag{13}$$

which is the desired result (5)

■

2. A digital decimation stage consists of a linear filter and the down-sampling step. The real signal to be decimated has the following characteristics:

   - $\min_f |X(f)| = A, \quad |f| < 0.2$
   - $|X(f)| = 0, \quad 0.2 < |f| < 0.3$
   - $\max_f |X(f)| = B, \quad 0.3 < |f| < 0.5$

   where $B/A = 1/2$. The low frequency portion should be kept after decimation and the high frequency portion should be rejected.

   a) Derive a down-sampling factor D such that the spectral information between $f = 0$ and $f = 0.2$ can be preserved. (2p)

   b) Derive a filter specification such that the SNR in the down-sampled signal is at least 40 dB and the maximal passband deviation is maximally 3 dB. (2pt)

   c) An Nth order Butterworth filter with 3 dB cut off at Ω_c has an amplitude characteristic of the form

   $$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}}$$

   Derive the minimal order digital sampled IIR filter of Butterworth type which meets the specifications above. Use the Bilinear design method where the relevant equations are

   $$s = 2\frac{z-1}{z+1} \quad \text{and} \quad \Omega = 2\tan\frac{\omega}{2}$$

   In your solution you should derive the minimal order and illustrate the steps needed to derive the digital filter. You do not have to calculate the filter coefficients. (6pt)

   Solution:

   (a) Since the bandwidth of the real signal to keep is is 0.2 and positioned between $f = 0$ and $f = 0.2$ the requirement on the integer D is $0.2D \leq 0.5$ which yield $D = 2$.

   (b) After the downsampling a factor 2 the power in the frequency band between 0.25 and 0.5 will create aliasing distortion. To keep this distortion below 40 dB (relative to the signal to keep) we need a low-pass filter to attenuate the signal between 0.3 an 0.5 a factor of $10^{40/20}/2 = 50$. The division of 2 originates from the fact that B/A=1/2. Hence the LP filter need to have a stop-band attenuation larger than 50 times between 0.3 and 0.5 which yields a stopband edge frequency of $f_s = 0.3$. Since the signal in the frequency band 0 to 0.2 should be in the passband we set the passband edge frequency f_p to 0.2. In summary we need a LP-filter with a passband deviation of maximally 3 dB with $f_p = 0.2$ and the stop band edge frequency $f_s = 0.3$ and a filter gain in the stop band less than 1/50.

(c) We translate the specifications to continuous time frequencies $\Omega_p = 2\tan\frac{2\pi 0.2}{2}$ and $\Omega_s = 2\tan\frac{2\pi 0.3}{2}$. Using the amplitude expression for the filter we obtain the equation

$$\left|\frac{1}{50}\right|^2 \leq \frac{1}{1+(\Omega_s/\Omega_p)^{2N}} = \frac{1}{1+(\tan(0.3\pi)/\tan(0.2\pi))^{2N}}$$

which yields

$$N \geq \frac{1}{2}\frac{\log(2500-1)}{\log(\tan(0.3\pi)/\tan(0.2\pi))} = 6.12$$

The required minimal order is thus 7. With the passband edge frequency $\Omega_c = 2\tan(0.2\pi)$ the continuous time Butterworth filter is derived with transfer function $H(s)$. The discrete time filter is then obtained by employing the bilinear transform $s = 2\frac{z-1}{z+1}$.

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3. In this problem an LMS-filtering setup, similar to the one used in the second project is used. The output of the LMS filter is $\hat{x} = y * h$, (here $*$ denotes convolution) and the filtering error which is minimized by the adaptation is is $e = x - \hat{x}$.

Consider the following scenarios and, for each scenario, derive en expression of what h will converge to and what the error signal e become after convergence. The impulse responses g and f are FIR filters and the signals s, w, w_1, w_2 are independent of each other and each of them are zero mean random processes with a constant power spectrum (a white noise signal). Assume that the filter is tuned to converge to the optimal filter and the length of the LMS-filter h is at least as long as the length of $f * g$.

(a) $y = w$ and $x = s + 2w * g$ (3pt)

(b) $y = w_1$ and $x = g * (s + w_1 + w_2)$ (3pt)

(c) $y = w$, $x = g * (s + f * w)$ (3pt)

(d) $s_1 = w_2 + f * w_1$, $y = w_1$ and $x = g * (s_1 + w_1)$ (3pt)

Solution: The filter will converge towards the filter which minimizes the variance of the error $\mathbf{E}\, e^2(n)$. We obtain the solution by investigating this variance for the four different cases

(a) We obtain

$$\mathbf{E}\, e^2 = \mathbf{E}(x - y * h)^2 = \mathbf{E}(s + 2w * g - w * h)^2 = \mathbf{E}\, s^2 + \mathbf{E}((2g - h) * w)^2$$

where the last equality follows from the fact that the signals w and s are independent and zero mean. Clearly, the choice $h = 2g$ will minimize the error variance expression and finally $e = x - y * h = s + 2w * g - w * 2g = s$.

(b)

$$\mathbf{E}\, e^2 = \mathbf{E}(x - y * h)^2 = \mathbf{E}(g * (s + w_1 + w_2) - w_1 * h)^2 = \mathbf{E}(g * (s + w_2))^2 + \mathbf{E}((g - h) * w_1)^2$$

which is minimized for $h = g$. The error is then $e = g * (s + w_2)$.

(c)

$$\mathbf{E}\, e^2 = \mathbf{E}(x - y * h)^2 = \mathbf{E}(g * (s + f * w) - w * h)^2 = \mathbf{E}(g * s)^2 + \mathbf{E}((g * f - h) * w)^2$$

which is minimized for $h = g * f$. The error is then $e = g * s$.

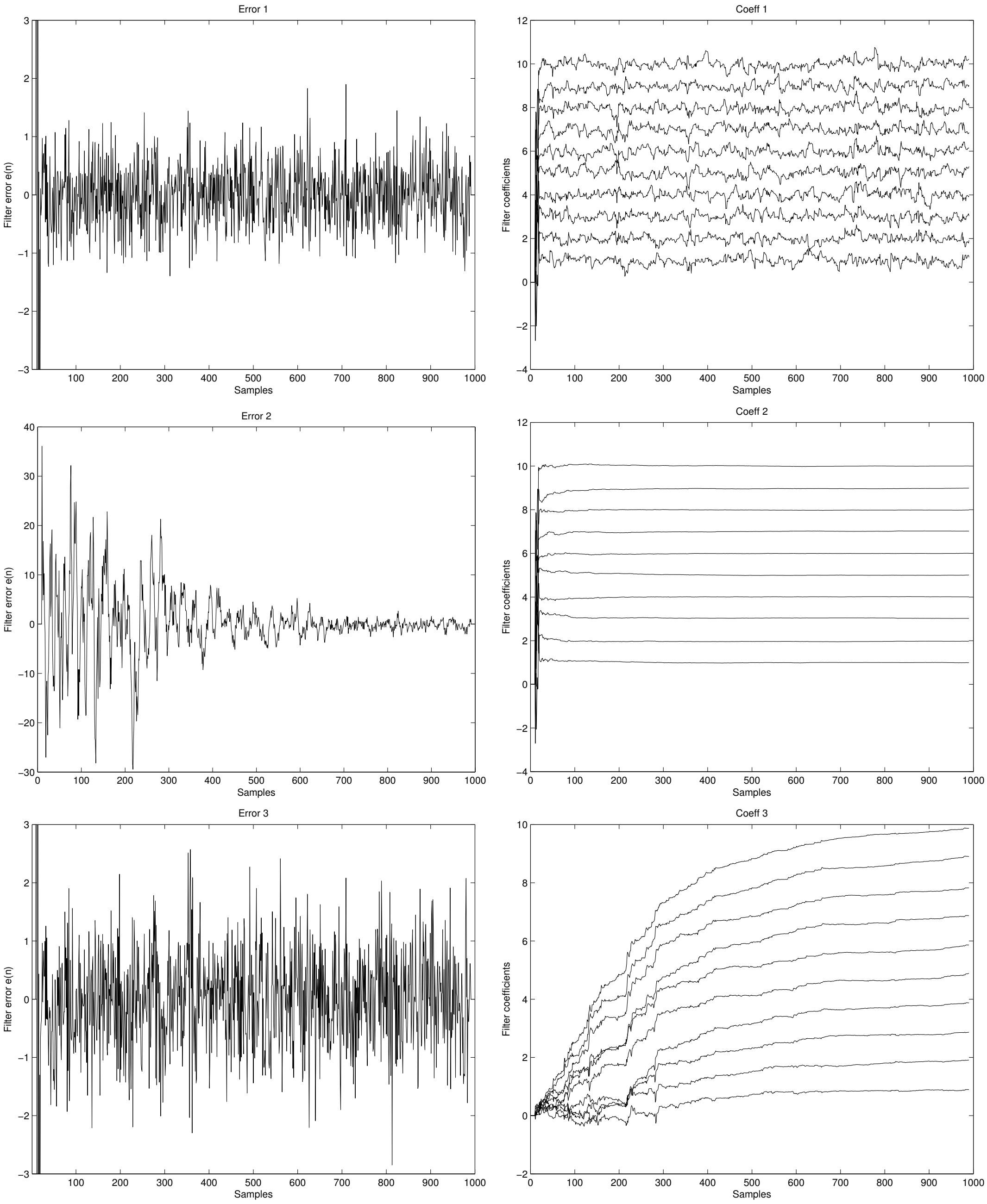

Figure 1: Graphs for adaptive filtering problem. Left column: Error e. Right column: Filter coefficients h. *Note that the error plots have different scales!*

(d)

$$\mathbf{E}\, e^2 = \mathbf{E}(x - y * h)^2 = \mathbf{E}(g * (w_1 + w_2 + f * w_1) - w_1 * h)^2 = \mathbf{E}(g * w_2)^2 + \mathbf{E}((g * (f+1) - h) * w_1)^2$$

which is minimized for $h = g * (f + 1)$. The error is then $e = g * w_2$.

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4. In Figure 1 you find 6 graphs. The 3 left graphs are the time evolution of the filtering error signal and the 3 right graphs correspond to the filter coefficient evolution over time. Below

you find the filter updating equations for three different adaptive filtering algorithms. Which of the error signal graphs and the filter coefficient graphs correspond to which algorithm respectively? Motivate your choices carefully. What are the names of the algorithms. (10pt)

A1)
```
Ri = 1000*eye(10);
alpha=1;
for k=Nh:1000-Nh,
  e(k) = x(k) - h(:,k)'*y(k-Nh+1:k);
  K = Ri*y(k-Nh+1:k);
  Ri = (Ri - K*K'/(alpha + y(k-Nh+1:k)'*K))/alpha;
  h(:,k+1) = h(:,k) + Ri*y(k-Nh+1:k)*e(k);
end
```

A2)
```
Ri = 0.5e-2;
for k=Nh:1000-Nh,
  e(k) = x(k) - h(:,k)'*y(k-Nh+1:k);
  h(:,k+1) = h(:,k) + Ri*y(k-Nh+1:k)*e(k);
end
```

A3)
```
Ri = 1000*eye(10);
alpha=0.8;
for k=Nh:1000-Nh,
  e(k) = x(k) - h(:,k)'*y(k-Nh+1:k);
  K = Ri*y(k-Nh+1:k);
  Ri = (Ri - K*K'/(alpha + y(k-Nh+1:k)'*K))/alpha;
  h(:,k+1) = h(:,k) + Ri*y(k-Nh+1:k)*e(k);
end
```

Solution: From the code we identify that A1 and A3 are similar. The difference is the variable `alpha` which is 1 for A1 and 0.8 for A3. A1 is recursive least-squares (RLS) and A3 is RLS with forgetting factor 0.8. A2 is least mean squares (LMS). RLS has superior convergence properties compared to LMS $\Rightarrow$ Coeff 3 is LMS=A2. The size of the error is closely related to the filter coefficients (and vice versa) hence $\Rightarrow$ Error 2 is A2. RLS with forgetting factor 0.8 will forget older values and hence will be more sensitive to measurement errors. Since Coeff 1 has larger variability in the end compared to Coeff 2 Coeff 1 belongs to A3 and consequently Coeff 2 belong to A1. With the same argument the errors are matched accordingly: Error 3 belong to A3 and Error 1 to A1.

Summary:

A1	Error 1	Coeff 2
A2	Error 2	Coeff 3
A3	Error 3	Coeff 1

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5. In the following subproblems you will get 1 point if the answer is correct and 1 point if the answer is correctly motivated.

   (a) The magnitude of the DTFT of two FIR filters designed with the Parks-McClellan optimal equiripple FIR filter design method are shown in Figure 2. Which of the two filters have the longest length. (2pt)

   (b) Two high-pass filters of the same length are designed using the window method. Filter A is designed using the window function A and filter B is designed with window function B. The DTFT of the two window functions are shown in Figure 3.

      i. Which of the two filters will have the best stop-band attenuation? (2pt)

      ii. Which of the two filters will have the largest transition band? (2pt)

   (c) The magnitude of the DTFT of the impulse response of a filter is shown in Figure 4.

      i. Describe the type of filter (HP,LP,BP or BS) (2pt)

ii. What is/are the crossover frequency/frequencies in Hertz if we assume the sample rate of the filter is 20 kHz. (2pt)

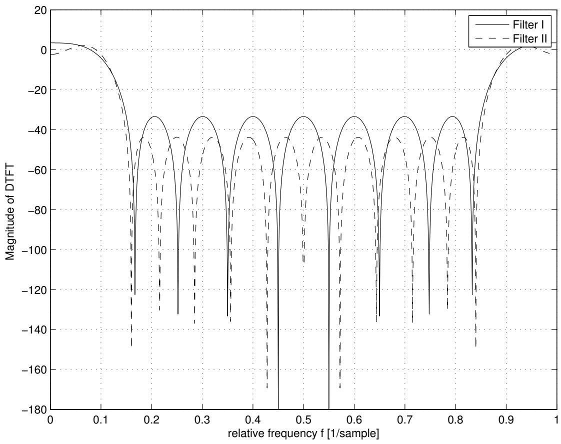

Figure 2: Magnitude DTFT of two FIR filters.

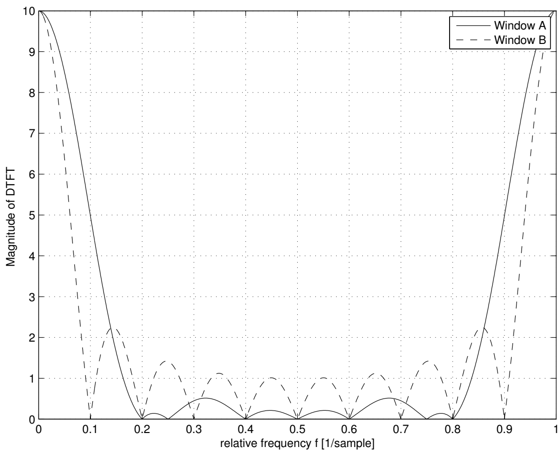

Figure 3: Magnitude DTFT of two window functions.

Solution:

(a) A longer filter means a more flexible filter. A flexible filter can be made to have better performance, for example better stop band attenuation. ⇒ Filter II is the longest.

(b) i. The frequency function for this design is the result of the frequency domain convolution between the ideal LP-filter and the DTFT of the window function. If the window has small side lobes ⇒ the filter will have a high stop band attenuation. Since Window A has the lowest side lobes ⇒ Filter A will have the best stop-band attenuation.

ii. Again due to the convolution the width of the transition band is proportional to the width of the main lobe of the window function. ⇒ Filter A will have the largest transition band.

(c) i. Since the DTFT is plotted in relative frequency where 1=sampling frequency and 0.5 is Nyquist frequency. Hence filter is a High-Pass filter (HP).

ii. The crossover frequency for the filter is at relative frequency 0.4 which in Hertz is $f_c = 0.4 * 20$ kHz $= 8$ kHz.

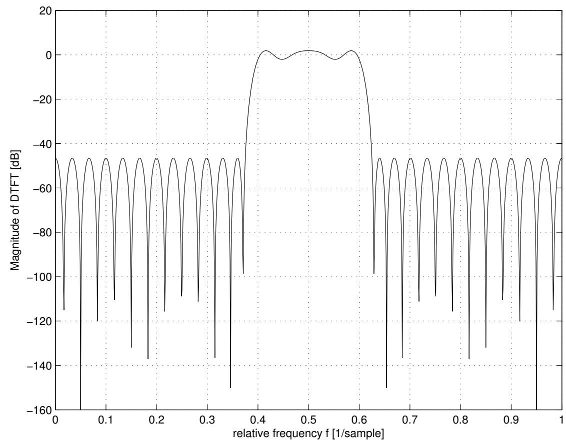

Figure 4: Magnitude DTFT of the impulse response of a filter.

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END - Good Luck!