Suggested Solutions to Examination
SSY130 Applied Signal Processing

14:00-18:00, April 8, 2010

Instructions

- *Responsible teacher:* Tomas McKelvey, ph 8061. Teacher will visit the site of examination at 14:45 and 16:00.

- Score from the written examination will together with course score determine the final grade according to the Course PM.

- Solutions are published on the course home-page latest 15 noon April 12.

- Your preliminary grade is reported to you via email.

- Exam grading review will be held in the "Blue Room" on level 6 at 12:15-12:45 on April 23, 2010.

Allowed aids at exam:

- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta).

- Any calculator

- One a4 size single page with written notes

Other important issues:

- The ordering of the questions is arbitrary.

- All solutions should be well motivated and clearly presented in order to render a full score. Unclear presentation or adding, for the problem in questing, irrelevant information render a reduction of the score.
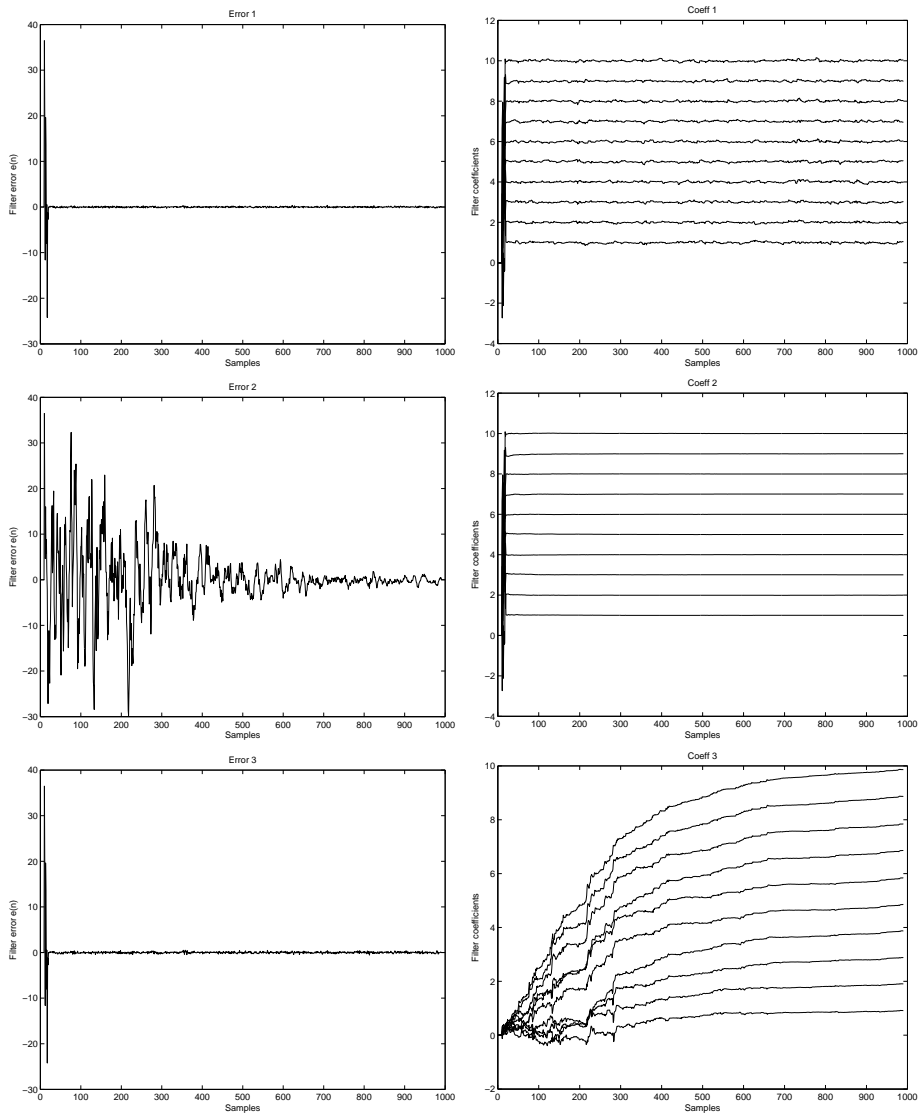
- The maximum score is 52 points.

Figure 1: Graphs for adaptive filtering problem. Left column error. Right column coefficients.

Problems

1. In Figure 1 you find 6 graphs. The 3 left graphs are the time evolution of the filtering error signal and the 3 right graphs correspond to the filter coefficient evolution over time. Below you find the filter updating equations for three different adaptive filtering algorithms. Which of the error signal graphs and the filter coefficient graphs correspond to which algorithm respectively? Motivate your choices carefully. What are the names of each of the algorithms. (10p)

A1)
```
Ri = 1000*eye(10);
alpha=1;
for k=Nh:1000-Nh,
  e(k) = d(k) - hx(:,k)'*x(k-Nh+1:k);
  K = Ri*x(k-Nh+1:k);
  Rinv = (Ri - K*K'/(alpha + x(k-Nh+1:k)'*K))/alpha;
  hx(:,k+1) = hx(:,k) + Ri*x(k-Nh+1:k)*e(k);
```

2

```
        end
```

A2)
```
    Ri = 0.5e-2;
    for k=Nh:1000-Nh,
      e(k) = d(k) - hx(:,k)'*x(k-Nh+1:k);
      hx(:,k+1) = hx(:,k) + Ri*x(k-Nh+1:k)*e(k);
    end
```

A3)
```
    Ri = 1000*eye(10);
    alpha=0.8;
    for k=Nh:1000-Nh,
      e(k) = d(k) - hx(:,k)'*x(k-Nh+1:k);
      K = Ri*x(k-Nh+1:k);
      Rinv = (Ri - K*K'/(1 + x(k-Nh+1:k)'*K))/alpha;
      hx(:,k+1) = hx(:,k) + Ri*x(k-Nh+1:k)*e(k);
    end
```

Solution: A1 is RLS, A2 is LMS and A3 is RLS with forgetting factor 0.8. RLS is superior to LMS in convergence so clearly Error 2 and Coeff 3 belong to the LMS algorithm. The difference between RLS with and without forgetting is that the RLS without forgetting factor will converge as the number of samples increases even in presence of noise. Since the Coeff 1 graph has some residual variations around the correct values this graph belongs to algorithm A3. Since the coefficients vary for A3 this also leads to a higher residual variance for the error. Graph Error 1 has less residual variance as compared to Error 3 and hence Error 1 belongs to RLS without forgetting. In conclusion:

- A1 Error 1 Coeff 2
- A2 Error 2 Coeff 3
- A3 Error 3 Coeff 1

□

2. Each cylinder in a four-stroke combustion engine provides a net torque contribution once every second revolution, i.e. once for each 720 degrees of crank shaft revolution. In the top graph in the figure below, 720 samples of the signal from a crankshaft mounted torque sensor is shown. The signal has been sampled every one degree of crankshaft revolution. In the bottom graph in the figure the 41 first bins of the 720 samples DFT are shown.
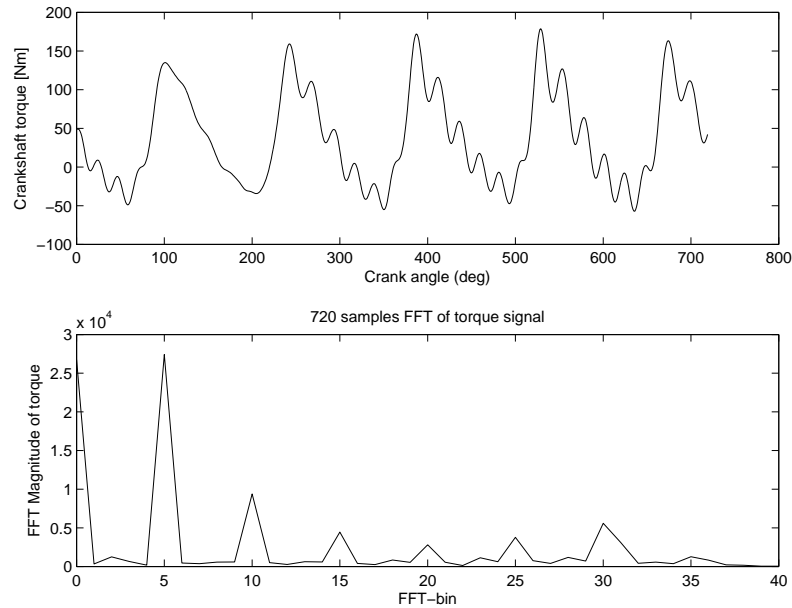
(a) How many cylinders does the engine have? (3p)

(b) During the measurement period the engine speed variations were very small. The average sample rate was 5879 samples per second. What was the speed of the engine given in RPM (revolutions per minute)? (2p)

(c) The sampling of the signals in this application is not driven by a clock but instead of an angle encoder. Discuss some signal processing challenges which arise when sampling is not based on a clock. (5p)

Solution:

(a) After 720 degrees of crankshaft revolution the engine has made a full cycle and since the graph covers 720 degrees and we see 5 pulses in the torque signal we conclude that the engine has 5 cylinders. In the FFT graph we also se that the first bin with a large energy is at bin 5 (5 per 720 degrees).

(b) Each sample corresponds to 1 degree of crankshaft revolution, 1 revolution is 360 degrees and one minute is 60 seconds. Hence RPM $= \frac{5879/360}{*} 60 = 980$

(c) An engine in an automotive application is operating at varying speeds, e.g. 800 RPM - 6000 RPM. With a sampling based on crankshaft angles this means that the sampling freuqency is varying over a large range. Anti aliasing filter design is then a challenge since the Nyquist frequency is varying with the engine speed and the choice of (a fixed) cut off frequency has to be a trade-off between bandwidth of the measurements and aliasing effects. Discrete time filtering of a signal which has been sampled in a time varying fashion also requires special considerations particularly time varying filters need to be employed.

∎

3. A digital signal processing system is operating at a sampling frequency of 10 kHz. A linear phase high-pass FIR filter with unit gain in the passband and a cut-off frequency of 3 kHz is to be designed.

(a) Determine the cut-off frequency in the unit of radians per sample. (2pt)

(b) Design a causal FIR filter with 5 filter coefficients ($N = 5$) using the Fourier series method with the specifications as above. Hint: The inverse Fourier transform for a symmetric frequency function is given by

$$c_n = \frac{1}{\pi} \int_0^{\pi} H_D(\omega)\cos(n\omega)d\omega$$

where ω is in radians per sample. (5p)

(c) A so called window function, other than the implicit rectangular window, can be used to modify the designed filter. Explain why this technique is often employed and the effects (both positive and negative) it has on the designed filter. (4p)

(d) Use the window (a Hanning window) defined by

$$w_n = \sin^2((n+1)\pi/(N+1)); \quad 0 \le n < N$$

and modify your filter design. (2pt)

(e) Since the filter is a high-pass filter it is natural that the amplitude function at half the sampling frequency (the Nyquist frequency) is equal to one. Determine the amplitude at the Nyquist frequency for your filter design. (2pt)

(f) Explain any deviation from amplitude one and suggest how to modify the filter to obtain amplitude one at the Nyquist frequency (3pt)

Solution:

(a) $\omega_c = \frac{2\pi f_c}{f_s} = \frac{2\pi * 3}{10} = \frac{3}{5}\pi = 1.88$ rad/s

(b) Symmetric coefficients around $n = 0$ are obtained by inverse Fourier transform of the desired frequency response :

$$c_n = \frac{1}{\pi}\int_{\omega_c}^{\pi}\cos(n\omega)d\omega = \frac{1}{\pi n}\left[\sin(n\omega)\right]_{\omega_c}^{\pi} = \frac{1}{\pi n}(\sin(\pi n) - \sin(n\omega_c)) \quad n = 0, \pm1, \pm2$$

which yields

$$c_n = \begin{cases} 0.4 & n = 0 \\ -0.303 & n = \pm1 \\ 0.094 & n = \pm2 \end{cases}$$

This impulse response is non-causal and needs to be delayed 2 samples to yield a causal filter.

$$\begin{bmatrix} h_0 & h_1 & h_2 & h_3 & h_4 \end{bmatrix} = \begin{bmatrix} 0.094 & -0.303 & 0.4 & -0.303 & 0.094 \end{bmatrix}$$

(c) The window method starts with a brick-wall design with an (implicit) infinite filter length. The truncation to a finite length filter leads to two major effects; 1) the transition band between the stop band and pass band becomes wider and 2) ripple in the stop band and pass band appears which limits the stop-band attenuation as well as distorts the constant gain in the passband. The relative effect of these two properties can be controlled by multiplying the filter coefficients with different window functions.

(d) The window is applied by element-wise multiplication $h_i^w = h_i w_i$:

$$\begin{bmatrix} h_0^w & h_1^w & h_2^w & h_3^w & h_4^w \end{bmatrix} = \begin{bmatrix} 0.023 & -0.227 & 0.4 & -0.227 & 0.023 \end{bmatrix}$$

(e) $|H^w(\pi)| = |\sum_{k=0}^{4} h_k^w e^{-jk\pi}| = \sum_{k=0}^{4} h_k^w(-1)^k = 0.901$

(f) The deviation at the Nyquist frequency has it's origin both in the truncation of the impulse response to a finite length and the effect of the multiplication with the Hanning window. Since the transfer function is a linear function of the filter coefficients a rescaling yields the desired result $h_i^f = h_i^w/|H^w(\pi)|$

$$\begin{bmatrix} h_0^f & h_1^f & h_2^f & h_3^f & h_4^f \end{bmatrix} = \begin{bmatrix} 0.026 & -0.252 & 0.444 & -0.252 & 0.026 \end{bmatrix}$$

4. In a signal quality monitoring system it is desired to monitor the amplitudes of the fundamental frequency of 50 Hz and the amplitude of the first harmonic with a frequency of 100 Hz. In the application amplitudes and phases are varying over time at a rate slower than the fundamental frequency. Your task is to design a Kalman filtering based solution which estimates and track both amplitudes of both signal components. We assume the signal measured can be described as

$$y(n) = A \sin(2\pi\,50\,n\Delta t + \varphi_A) + B \sin(2\pi\,100\,n\Delta t + \varphi_B) + e(n)$$

where $e(n)$ is zero mean white noise with a Gaussian distribution and Δt is the sampling interval.

(a) The amplitudes appear linear in the signal model but the phases are non-linear which makes a direct application of the Kalman filter not possible. Show that the alternate model

$$y(n) = \alpha \sin(2\pi\,50\,n\Delta t) + \beta \cos(2\pi\,50\,n\Delta t) + \gamma \sin(2\pi\,100\,n\Delta t) + \theta \cos(2\pi\,100\,n\Delta t)$$

is an equivalent model and how the amplitudes A and B can be derived from the knowledge of α, β, γ and θ. (4p)

(b) Rewrite the signal model into a state-space form suitable for a direct Kalman filter application. (5p)

(c) It is known that the amplitude of the 100 Hz signal component changes at a much slower rate compared to the 50 Hz component. Explain how this knowledge can be used when tuning the Kalman filter. (5p).

Solution:

(a) Defining $A = \sqrt{\alpha^2 + \beta^2}$ and $\cos \varphi_A = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}$ we obtain

$$A \sin(t + \varphi_A) = \alpha \sin t + \beta \cos t$$

and the result follows.

(b) The parameters to track are α, β, γ and θ and we hence put them into the state-vector

$$x(n) = \begin{bmatrix} \alpha & \beta & \gamma & \theta \end{bmatrix}^T$$

From the problem formulation the amplitudes are assumed to vary slowly and a simple random walk model is suitable for the state-update equation:

$$x(n+1) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} x(n) + w(n)$$

where $w(n)$ is a zero mean random vector with covariance matrix Q_1. The measurement equation is then derive d directly from the signal model as

$$y(n) = C_n x(n) + e(n)$$

where the time varying row vector C_n is given by

$$C_n = \begin{bmatrix} \sin(2\pi\,50\,n\Delta t) & \cos(2\pi\,50\,n\Delta t) & \sin(2\pi\,100\,n\Delta t) & \cos(2\pi\,100\,n\Delta t) \end{bmatrix}$$

and the measurement noise $e(n)$ is zero mean and a (co)variance Q_2.

6

(c) In the model the rate of change for each of the parameters are described in the diagonal elements of Q_1. To capture the knowledge that the 100 Hz signal is changing amplitudes at a rate much slower then the 50 Hz signal we can for example use

$$Q_1 = \begin{bmatrix} s_1 I_2 & 0 \\ 0 & s_2 I_2 \end{bmatrix}$$

where I_2 are identity matrices of size 2 by 2 and s_1 and s_2 are two positive scalars. By setting $s_1 > s_2$ we explicitly model that α and β varies faster than the two other parameters. The relative sizes between on one hand s_1 and s_2 and on the other hand Q_2 control trade off between the how fast the filter can track changes and the variability of the estimates of the tracked parameters. If Q_2 is selected small in comparison with s_1 and s_2 then measurements are assumed "good" and tracking speed is fast.

∎