Examination
SSY130 Applied Signal Processing

14:00-18:00, December 15, 2009

Instructions

- *Responsible teacher:* Ingemar Andersson, ph 1784. Teacher will visit the site of examination at 14:45 and 16:00.
- Score from the written examination will together with course score determine the final grade according to the Course PM.
- Solutions are published on the course home-page latest 12 noon December 21.
- Your preliminary grade is reported to you via email.
- Exam grading review will be held in the "Blue Room" on level 6 at 12:30-13:00 on January 20, 2010.

Allowed aids at exam:

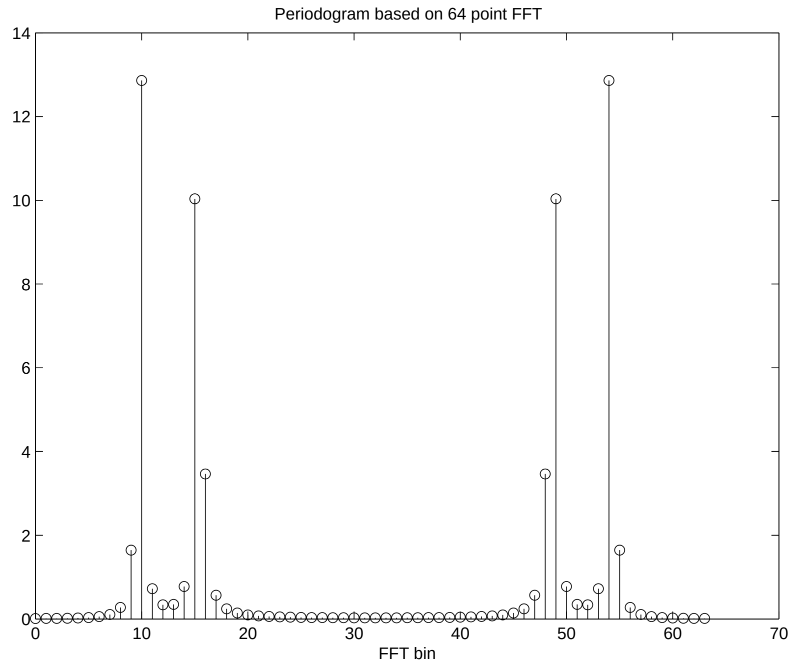
- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta).
- Any calculator
- One a4 size single page with written notes

Other important issues:

- The ordering of the questions is arbitrary.
- All solutions should be well motivated and clearly presented in order to render a full score. Unclear presentation or adding, for the problem in questing, irrelevant information render a reduction of the score.
- The maximum score is 52 points.

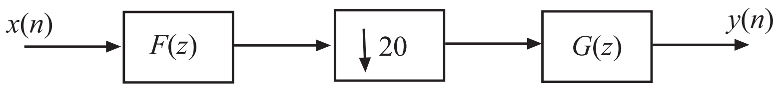
Problems

1. A signal consists of two sinusoidal components with frequencies, f_1 and f_2. It is known that $1 < f_1 < 2$ [Hz] and $7 < f_2 < 8$ [Hz]. The signal is sampled at the rate 10 Hz, without using any anti-aliasing filter. A 64-point FFT is computed from the digital signal, and based on this the periodogram is formed. The result is shown in the figure below.



Based on the plot, determine (if possible) approximate values of f_1 and f_2 (in Hz). (4p)

2. A lowpass filter with passband edge frequency $f_p = 200$ Hz, and stopband edge frequency $f_s = 210$ Hz is desired (from x(n) to y(n)). The sampling rate is 44.1 kHz for the input signal $x(n)$. To simplify the implementation, a decimation of a factor of 20 is proposed, according to the figure below:



Determine suitable specifications for the "anti-aliasing" filter $F(z)$! Only the edge frequencies f_p and f_s are required for the filter $F(z)$. (3p)

3. An engineer is monitoring a slowly varying physical value by sampling it. He notices a large variation between consecutive samples and tries to remove the variation by caluclating a running average

$$\bar{x}(n) = \frac{1}{N} \sum_{l=n}^{n-N+1} x(l).$$

where $x(n)$ is the sampled signal and $\bar{x}(n)$ is the running average. To test how the running averaging works he uses a sinusodal signal as a test input and records the magnitude of the running average. The magnitude as a function of the frequency of the input signal is shown in Figure 1. The unit on the frequency axis is normalized with respect to the sampling frequency.

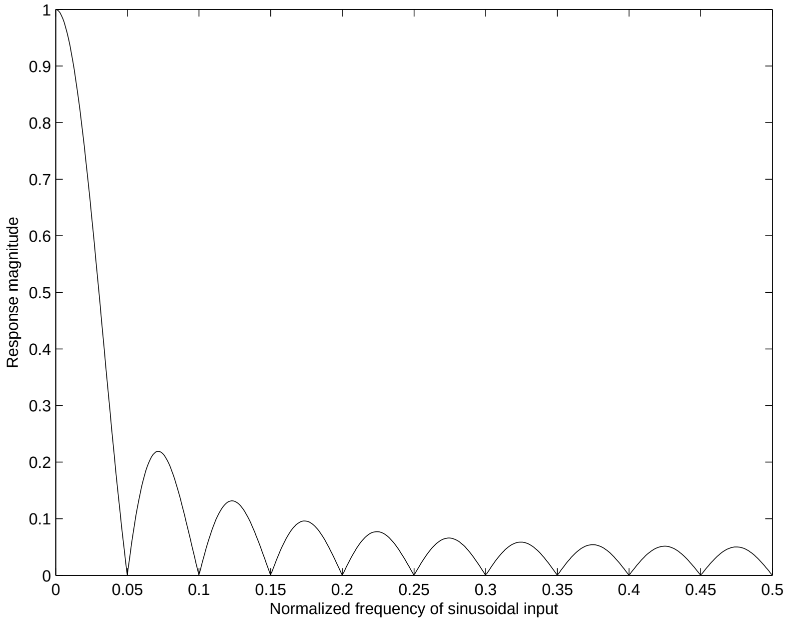


Figure 1: Magnitude of running average signal as a function of frequency.

(a) What is the length N of the averaging operation. (5pt)

(b) Suggest a length N and sampling frequency such that a 50 Hz disurbance is completely cancelled by the averaging operation. Is the solution unique? What characterizes a valid solution? (5pt)

4. An LMS adaptive filter with a FIR structure of length 5 is used for a system modelling application, i.e. where $y(n)$ is the input to the unknown system and $x(n)$ is output of the system. The algorithm is tested on the same unknown system with four different input signals.

   (1) $y(n) = \sin(\omega_0 n)$

   (2) $y(n) = \sin(2\omega_0 n)$

   (3) $y(n) = Ae(n)$

   (4) $y(n) = 2Ae(n)$

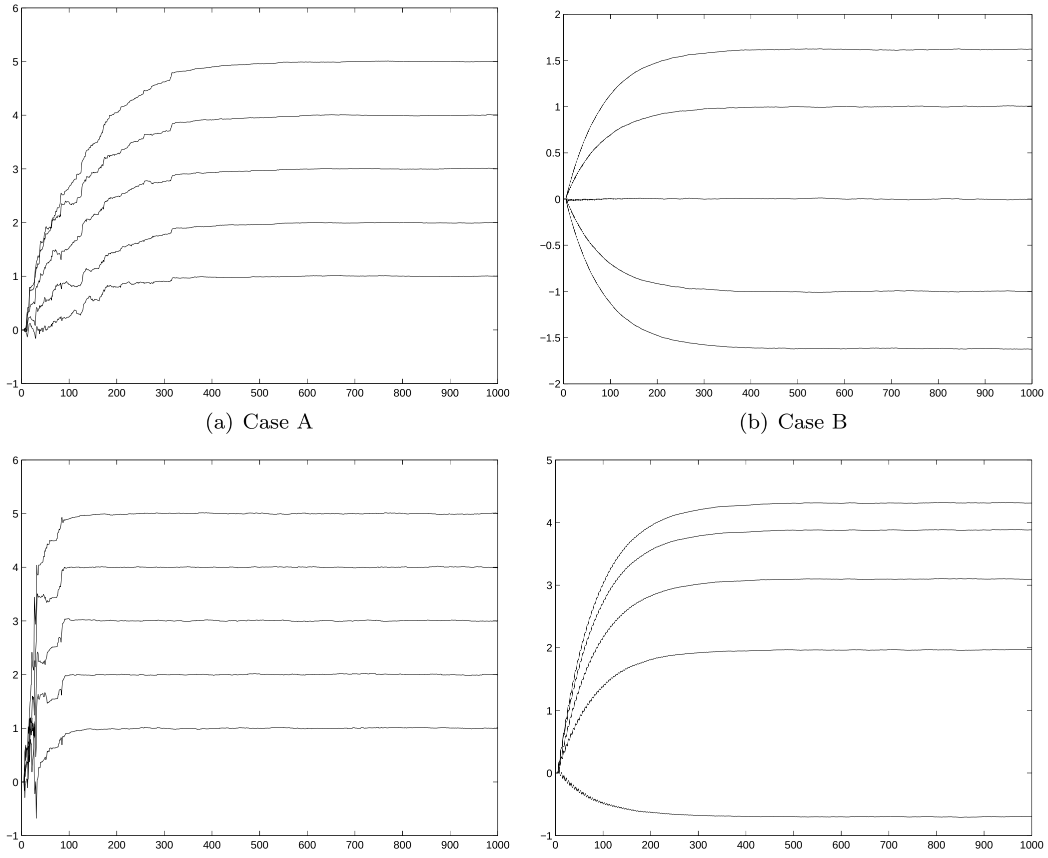
   where $e(n)$ is a zero mean, independen over time, stochastic process and $\omega_0 = 2\pi/10$. In all cases an identical constant step length μ was used. In Figure 2 the evolution over time of the estimated filter coefficents are shown for the four cases. In Figure 3 the magnitude of the frequency response is shown for the filters defined by the final values of the estimated coefficients. For each of the input signals explain which of the subplots in Figure 2 and which of the transfer functions in Figure fig:3 corresponds to the specific input signal $y(n)$. (10pt)

5. A canonical form implementation of a second order IIR filter can be described with the following three equations

$$\begin{aligned} s_1(n+1) &= x(n) - a_1 s_1(n) - a_2 s_2(n) \\ s_2(n+1) &= s_1(n) \\ y(n) &= b_0 x(n) + (b_1 - b_0 a_1) s_1(n) + (b_2 - b_0 a_2) s_2(n) \end{aligned}$$

   The "standard" form implementation of a second order IIR filter is

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$



(a) Case A

(b) Case B

(c) Case C

(d) Case D

Figure 2: Filter coefficient evolution over time

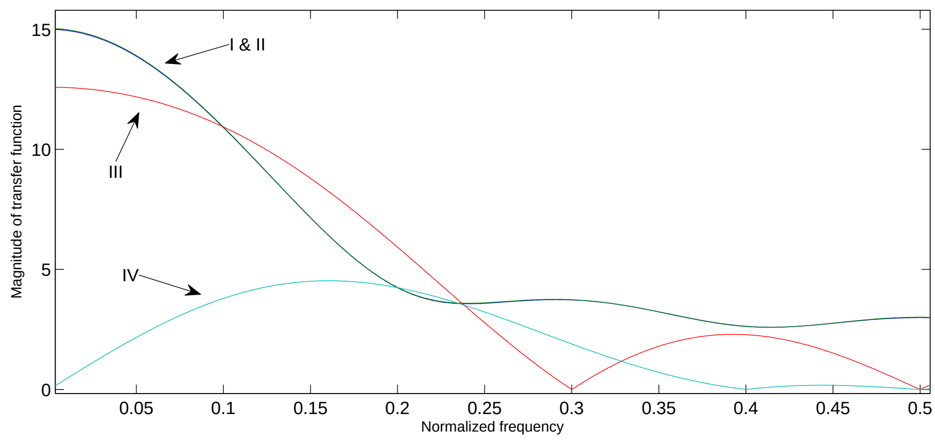


Figure 3: Magnitude of transfer function of the final filter for the four cases.

(a) Show that the two forms are identical from an input to output perspective. (Hint: Use the properties of the Fourier transform to elliminate the state-variables s_1 and s_2) (7pt)

(b) What benefit from an implementation perspective does the canonical form have as compared to the "standard form". (3pt)

6. The goal of this problem is to equalize a communication channel using a FIR Wiener filter. Assume the desired signal $x(n)$ is described as

$$x(n) - 0.8x(n-1) = e(n)$$

where $e(n)$ is zero-mean white noise with variance $\sigma_e^2 = 1$. The measured signal $y(n)$ is given by

$$y(n) = x(n) - 0.5x(n-1) + v(n)$$

where $v(n)$ is a zero-mean white noise with variance $\sigma_v^2 = 0.1$. Determine a filter

$$\hat{x}(n) = w_0 y(n) + w_1 y(n-1)$$

such that $E\left[(x(n) - \hat{x}(n))^2\right]$ is minimized! (10p)

7. Assume a stochastic process is formed by filtering white noise through a causal FIR filter of length N with impulse response $h(k)$. That is

$$y(n) = \sum_{k=0}^{N-1} h(k)e(n-k)$$

where $\mathbf{E}[e(n)] = 0$ and $\mathbf{E}[e(n)e(m)] = \sigma^2$ if $n = m$ otherwise 0. Derive the following quantities:

   (a) The expected value of the output $\mathbf{E}[y(n)]$ (1pt)

   (b) The power spectrum of the output $S_{yy}(\omega)$ (4pt)

END

Good Luck!

Suggested solutions to examination SSY130 Applied Signal Processing

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Solutions

1. The sampling leads to folding of the signal component with frequency f_2. The location of the folded frequency will then be between $10-8=2$ and $10-7=3$ [Hz]. The two leftmost peaks are at indices 10 and 15 which then imply the frequency $f_1 = 10/64 * 10 = 1.6$ [Hz] and frequency $15/64 * 10 = 2.3$ [Hz]. Hence the second peak is the folded frequency and consequently $f_2 = 10 - 2.3 = 7.7$ [Hz].

2. The key purpose of $F(z)$ is to mitigate aliasing distortion in the frequency band 0-210 Hz due to the downsampling operation. A low pass filter $F(z)$ with stop band frequency at $44100/20 - 210 = 1995$ would then provide high attenuation of the aliasing distortion occuring between $0 - 210$ Hz during the downsampling stage. Left is to define the passband edge frequency which we can set as low as 200 Hz to obatain as large transition region as possible (for low complexity filter design).

3. (a) The filter is a FIR filter with N coefficients. The frequency function of an averaging filter of length N is

   $$|H(\omega)| = \left|1/N \sum_{k=0}^{N-1} e^{j\omega k}\right| = \left|\frac{1-e^{j\omega N}}{N(1-e^{j\omega})}\right| = \frac{\sin(\omega N/2)}{N\sin(\omega/2)}$$

   First zero thus appear when $\omega N/2 = \pi$ which imply $fN = 1$. Since first zero is at $f = 0.05$ we obtain $N = 20$.

   (b) Clearly we have zeros when $fN = k$ and integers $k = 1, 2, \ldots, \text{floor}(N/2)$. Hence all solutions are given by $50N/f_s = k$ where f_s is the sampling frequency in [Hz].

4. Signals (3) and (4) are white noise with different amplitudes. White noise have a full rank autocorrelation matrix and the filter should converge to the vicinity of the true system paramters. Hence, diagrams A and C are the correct diagrams since they have the same converegene points. Since (4) has higher amplitude than (3) diagram C belongs to signal (4) and (3) to diagram A. In the transfer function diagram (3) and (4) belong to graph I and II (basically the same). Signals (1) and (2) only yields excitation at a single frequency and only one point in the transfer function will be correctly estimated. Hence, signal (1) belong to graph III and signal (2) belong to graph IV. The magnitude of the sum of the filter coefficents is the DC gain of a filter and consequently signal (1) and diagram D belong and signal (2) and diagram B. In summary we have

   (1) - D - III
   (2) - B - IV
   (3) - A - I & II
   (4) - C - I & II

5. (a) Use the notion that $x(n+1) = zx(n)$. Hence

$$\begin{aligned} zs_1(n) &= x(n) - a_1 s_1(n) - a_2 s_2(n) \\ zs_2(n) &= s_1(n) \\ y(n) &= b_0 x(n) + (b_1 - b_0 a_1) s_1(n) + (b_2 - b_0 a_2) s_2(n) \end{aligned}$$

Elimination of s_1 yields

$$\begin{aligned} z^2 s_2(n) &= x(n) - a_1 z s_2(n) - a_2 s_2(n) \\ &\Rightarrow \\ s_2(n) &= \frac{x(n)}{z^2 + a_1 z + a_2} \\ &\Rightarrow \\ y(n) &= b_0 x(n) + (b_1 - b_0 a_1) z s_2(n) + (b_2 - b_0 a_2) s_2(n) \\ &\Rightarrow \\ (z^2 + a_1 z + a_2) y(n) &= (b_0 z^2 + b_1 z + b_2) x(n) \end{aligned}$$

which is the desired result.

(b) The canonical form has the implementatin advantage that only two memory registers are required to implememnt an IIR filter of order 2.

6. Multiplying the equation for $x(n)$ with $x(n-1)$ and taking expectation yields $\phi_{xx}(1) = 0.8\phi_{xx}(0)$ and similarly using $x(n-2)$ yields $\phi_{xx}(2) = 0.8\phi_{xx}(1)$. The variance is obtained by squaring the defining equation which yeilds $\phi_{xx}(0) = 1/(1-0.8^2) = 2.78$.

Crosscorrelation calculations yield $\phi_{yx}(0) = \phi_{xx}(0) - 0.5\phi_{xx}(1) = 0.6\phi_{xx}(0)$ and $\phi_{yx}(1) = \phi_{xx}(1) - 0.5\phi_{xx}(2) = 0.48\phi_{xx}(0)$.

Finally the autocorrelation function for $y(n)$ are $\phi_{yy}(0) = 0.45\phi_{xx}(0) + 0.1$ and $\phi_{yy}(1) = 0.18\phi_{xx}(0)$.

Defining $\mathbf{w}^T = [\, w_0 \; w_1 \,]$ and $\mathbf{y(n)}^T = [\, y(n) \; y(n-1) \,]$ The variance of the prediction error is

$$\mathbf{E}(x(n) - \hat{x}(n))^2 = \mathbf{E}(x(n) - \mathbf{w}^T \mathbf{y}(n)))^2 = \phi_{xx}(0) - 2\mathbf{w}^T \Phi_{yx} + \mathbf{w}^T \Phi_{yy} \mathbf{w}$$

where

$$\Phi_{yy} = \mathbf{E}\begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} \begin{bmatrix} y(n) & y(n-1) \end{bmatrix} \quad \text{and} \quad \Phi_{yx} = \mathbf{E}\begin{bmatrix} y(n) \\ y(n-1) \end{bmatrix} x(n)$$

The stationary point of the variance expression is obtained by differentiation with respect to $\mathbf{w}$ and equating to zero.

$$-2\Phi_{yx} + 2\Phi_{yy}\mathbf{w} = 0$$

The variance optimal Wiener filter is given by

$$\begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \Phi_{yy}^{-1}\Phi_{yx} = \begin{bmatrix} 0.45 + 0.1/2.78 & 0.18 \\ 0.18 & 0.45 + 0.1/2.78 \end{bmatrix}^{-1} \begin{bmatrix} 0.6 \\ 0.48 \end{bmatrix} = \begin{bmatrix} 1.01 \\ 0.61 \end{bmatrix}$$

7. (a)

$$\mathbf{E}[y(n)] = \mathbf{E}[\sum_{k=0}^{N-1} h(k) e(n-k)] = \sum_{k=0}^{N-1} h(k)\, \mathbf{E}[e(n-k)] = 0$$

(b) The spectrum is defined as

$$S_{yy} = \sum_{\tau=-\infty}^{\infty} \phi_{yy}(\tau) e^{-j\omega\tau}$$

i.e., the DTFT of $\phi_{yy}(\tau)$ where

$$\phi_{yy}(\tau) = \mathbf{E}[y(n)y(n+\tau)] = \sum_k \sum_n h(k)h(m)\,\mathbf{E}[e(n-k)e(n+\tau-m)] = \sigma^2 \sum_m h(m-\tau)h(m)$$

We recognize that the right hand above is h convolved with the time reversed h. Hence the DTFT of $\phi_{yy}(\tau)$ is then the product of the DTFT of h and the DTFT of the time reversed h, i.e.,

$$S_{yy}(\omega) = \sigma^2 H(\omega)H(\omega)^* = \sigma^2 |H(\omega)|^2$$

END