Examination with solution suggestions
SSY130 Applied Signal Processing

Jan 8, 2008

Rules

Allowed aids at exam:

- L. Råde and B. Westergren, Mathematics Handbook (any edition, including the old editions called Beta).
- Any calculator
- One a4 size single page with written notes

The maximum score is 52 points. Correct solutions should be well motivated to render a full score.

Problems

1. In Figure 1 you find 6 graphs. The 3 left graphs are the time evolution of the filtering error signal and the 3 right graphs correspond to the filter coefficient evolution over time. Below you find the filter updating equations for three different adaptive filtering algorithms. Which of the error signal graphs and the filter coefficient graphs correspond to which algorithm respectively? Motivate your choices carefully. What are the names of each of the algorithms. (10 pt)

A1)
```
Ri = 1000*eye(10);
alpha=1;
for k=Nh:1000-Nh,
  e(k) = d(k) - hx(:,k)'*x(k-Nh+1:k);
  K = Ri*x(k-Nh+1:k);
  Ri = (Ri - K*K'/(alpha + x(k-Nh+1:k)'*K))/alpha;
  hx(:,k+1) = hx(:,k) + Ri*x(k-Nh+1:k)*e(k);
end
```

A2)
```
Ri = 0.5e-2;
for k=Nh:1000-Nh,
  e(k) = d(k) - hx(:,k)'*x(k-Nh+1:k);
  hx(:,k+1) = hx(:,k) + Ri*x(k-Nh+1:k)*e(k);
end
```

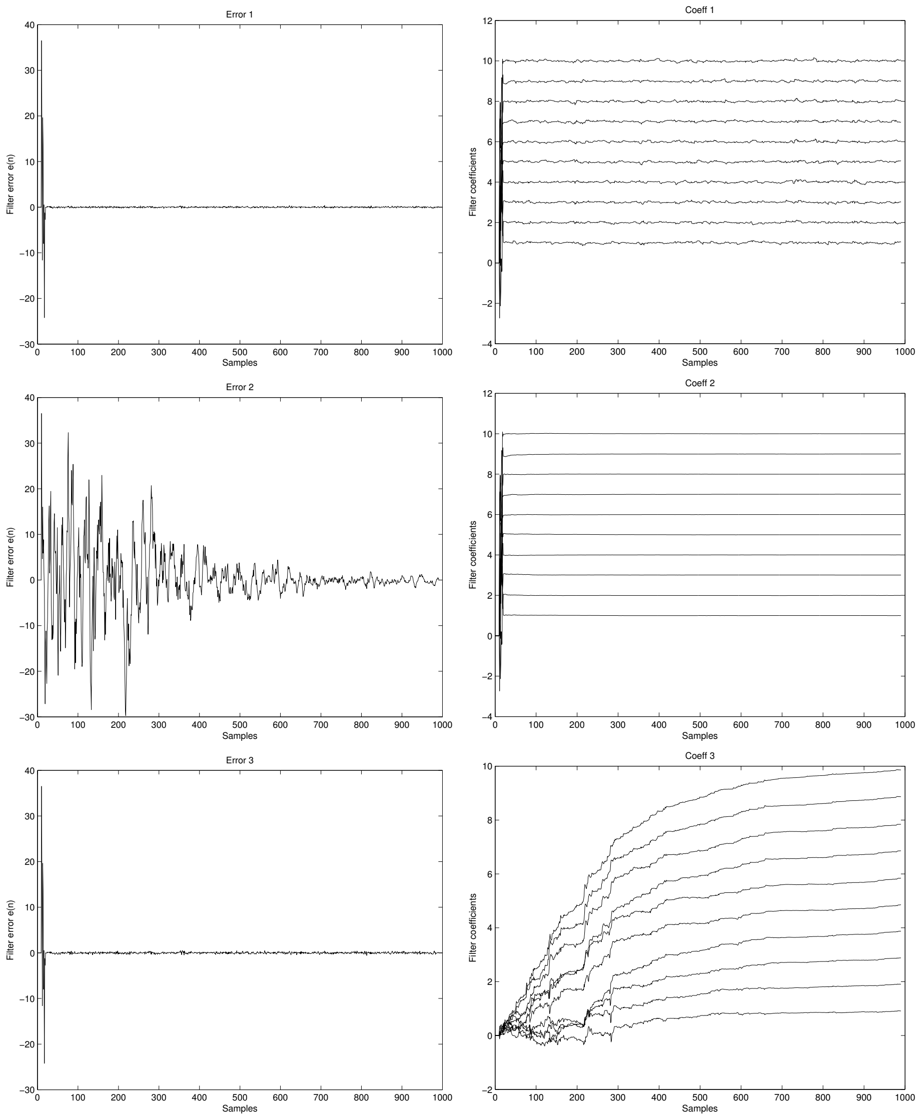

Figure 1: Graphs for adaptive filtering problem. Left column error. Right column coefficients.

A3)
```
Ri = 1000*eye(10);
alpha=0.8;
for k=Nh:1000-Nh,
  e(k) = d(k) - hx(:,k)'*x(k-Nh+1:k);
  K = Ri*x(k-Nh+1:k);
  Ri = (Ri - K*K'/(alpha + x(k-Nh+1:k)'*K))/alpha;
  hx(:,k+1) = hx(:,k) + Ri*x(k-Nh+1:k)*e(k);
end
```

Solution: A1 is RLS, A2 is LMS and A3 is RLS with forgetting factor 0.8. RLS is superior to LMS in convergence so clearly Error 2 and Coeff

3 belong to the LMS algorithm. The difference between RLS with and without forgetting is that the RLS without forgetting factor will converge as the number of samples increases even in presence of noise. Since the Coeff 1 graph has some residual variations around the correct values this graph belongs to algorithm A3. Since the coefficients vary for A3 this also leads to a higher residual variance for the error. Graph Error 1 has less residual variance as compared to Error 3 and hence Error 1 belongs to RLS without forgetting. In conclusion:

- A1 Error 1 Coeff 2
- A2 Error 2 Coeff 3
- A3 Error 3 Coeff 1

□

2. Consider a real signal of an odd length N with the following character:

$$x(n) = \begin{cases} x(N-1-n), & n = 0, \dots N-1 \\ x(n) = 0, & n < 0, n \geq N \end{cases} \tag{1}$$

Let $X(f)$ denote the discrete Fourier transform of the signal $x(n)$ and let $X(k)$ denote the N-point DFT of the signal. For each of the statements below, show if they are true or false

(a) $|X(f)| = |X(k)|$ where $f = 2\pi k/N$ and $0 < k < (N-1)/2$
(b) $X(f) = X(-f)^*$
(c) $\angle X(f) = \angle X(-f)$
(d) $X(k) = X((N-1)/2 + k), \quad k = 0, \dots (N-1)/2$
(e) $X(k) = -X(N-1-k), \quad k = 0, \dots, N-1$

(10p)

Solution: The signal $x(n)$ is symmetric and real but time shifted $M \triangleq (N-1)/2$ time steps. Define $x_s(n) = x(n+M)$ which is a signal symmetric around $n = 0$. Hence, we can write the Fourier transform as

$$\begin{aligned} X(f) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi f n} = e^{-j2\pi f M} \sum_{n=-M}^{M} x_d(n) e^{-j2\pi f n} \\ &= e^{-j2\pi f M} \sum_{n=-M}^{M} x_d(n) \cos(2\pi f n) \end{aligned} \tag{2}$$

where the last equality follows from the fact that x_d is real and symmetric. Now since $(e^{-j2\pi f M})^* = e^{-j2\pi(-f)M}$ statement b) is correct. Now $\angle X(f) = -2\pi f M \neq -2\pi(-f)M = \angle X(-f)$ so statement c) is incorrect. The discrete Fourier transform (DFT) is:

$$X(k) = \sum_{n=0}^{N-1} e^{-j2\pi k n/N} \tag{3}$$

DFT is thus samples at the Fourier transform at $f = k/N$. Hence statement a) is false. Due to the periodicity of $X(f)$ we have $X(f+1) = X(f)$. Using (2) this leads to $X(f) = X(-f)^* = X(1-f)^*$. This means that for the DFT we have $X(k) = X(N-k)^*$ for $k = 1, \ldots, M$ which shows that statements d) and e) are incorrect. $\square$

3. Consider the following setup

$$y(n) = \sum_{k=0}^{N_h-1} h(k)u(n-k) + v(n)$$

where $y(n)$ is a noisy measurement of the output of the FIR system, $u(n)$ is the input and $v(n)$ is the measurement noise. The measurement noise is modeled as a white stochastic process with zero mean and variance σ_v^2 and uncorrelated with the input signal. The input to the system is a periodic signal where the period is known to be N ($u(n) = u(n+N)$ for all n). Furthermore we also know that $N \geq N_h$.

(a) Write the input signal $u(n)$ as a discrete time Fourier series and show how the coefficients can be calculated from one period of $u(n)$. (2p)

(b) Present a method to calculate (estimate) the impulse response $h(k)$ from measurements of $y(n)$. (5p)

(c) What is the effect of the noise? Is the expected value of the calculated filter coefficients equal to the true ones? How does the variance of the estimate depend on the number of periods measured? (3pt)

Solution: (a) We start with the DFT relations

$$U(k) = \sum_{n=0}^{N-1} u(n)e^{-j2\pi kn/N} \quad \text{and} \quad u(n) = \frac{1}{N}\sum_{k=0}^{N-1} U(k)e^{j2\pi kn/N} \tag{4}$$

where the last term is the Fourier series representation of the periodic input signal.

(b,c) Since the input is periodic the output of the filter is periodic and the measurement is a periodic signal with white noise added. If we denote the DFT of one period of the measured signal $Y(k)$ clearly

$$Y(k) = H(k)U(k) + V(k) \tag{5}$$

where $H(k)$ is the N-point DFT of the impulse response of the FIR coefficients (zero padded to length N) and $V(k)$ is the DFT of the noise sequence. Since $v(k)$ is zero mean we have ($k, l = 0, \ldots, N-1$)

$$\begin{aligned} \mathbf{E}\{V(k)\} &= \mathbf{E}\sum_{n=0}^{N-1} v(n)e^{-j2\pi kn/N} = 0 \\ \mathbf{E}\{V(k)V(l)^*\} &= \begin{cases} N\sigma_v^2, & k = l \\ 0, & k \neq l \end{cases} \end{aligned} \tag{6}$$

where the last equality follows from the fact that

$$\sum_{k=0}^{N-1} e^{-j2\pi k/N} = \begin{cases} N, & k = 0, \pm N, \pm 2N, \dots \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

A natural estimate of the DFT of the FIR filter coefficients are thus

$$\hat{H}(k) = Y(k)/U(k) = H(k) + V(k)/U(k) \tag{8}$$

The expected value of this estimate is

$$\mathbf{E}\hat{H}(k) = H(k) \tag{9}$$

which means that the estimate is unbiased. The variance of the estimate is

$$\mathbf{E}\{|\hat{H}(k) - H(k)|^2\} = N\sigma_v^2/|U(k)|^2 \tag{10}$$

Here we see that the variance for frequency k is inversely proportional to the energy of the input signal at that frequency.

By the inverse DFT we finally obtain

$$\hat{h}(n) = \frac{1}{N}\sum_{k=0}^{N-1} \hat{H}(k)e^{j2\pi kn/N} = h(n) + \frac{1}{N}\sum_{k=0}^{N-1} V(k)/U(k)e^{j2\pi kn/N} \tag{11}$$

with expected value and variance

$$\begin{aligned} \mathbf{E}\{\hat{h}(n)\} &= h(n) \\ \mathbf{E}\{|\hat{h}(n) - h(n)|^2\} &= \frac{1}{N}\sum_{k=0}^{N-1} \sigma_v^2/|U(k)|^2 \end{aligned} \tag{12}$$

If we measure M periods we can average out the noise by simply forming the average over M periods

$$y_a(n) = \frac{1}{M}\sum_{m=0}^{M-1} y(n+mM), \quad n = 0, \dots, N-1 \tag{13}$$

and then use $y_a(n)$ in the method outlined above. The variance of the noise is thus reduced from σ_v^2 to σ_v^2/M and consequently also the variance of the filter coefficients estimates. □

4. A digital decimation stage consists of a linear filter and the down-sampling step. The real signal to be decimated has the following characteristics:

   - $\min_f |X(f)| = A, \quad |f| < 0.45/2$
   - $|X(f)| = 0, \quad 0.45/2 < |f| < 0.55/2$
   - $\max_f |X(f)| = B, \quad 0.55/2 < |f| < 1/2$

   where $B/A = 2$. The low frequency portion should be kept after decimation and the high frequency portion should be rejected.

a) What is a suitable down-sampling factor D? (2p)

b) Derive a filter specification such that the SNR in the down-sampled signal is at least 40 dB and the maximal passband deviation is maximally 3 dB. (2pt)

c) An N-th order Butterworth filter with 3 dB cut off at Ω_c has an amplitude characteristic of the form

$$|H(\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

Derive the minimal order digital sampled IIR filter of Butterworth type which meets the specifications above. Use the Bilinear design method where the relevant equations are

$$s = 2\frac{z-1}{z+1} \quad \text{and} \quad \Omega = 2\tan\frac{\omega}{2}$$

In your solution you should derive the minimal order and illustrate the steps needed to be performed to derive the digital filter. You do not have to calculate the filter coefficients. (6pt)

Solution: (a) The low frequency signal to retain occupy a little less than half the spectrum so a suitable factor to downsample is 2.

(b) The required SNR of 40 for the signal implies that we should have an alias effect for all frequencies which is less than 40 dB (100 in amplitude). The stated properties of the high an low frequency parts of the spectrum thus imply that the filter should attenuate the high frequency portion 200 times for all frequencies. If we define the -3 dB point of the filter as the crossover frequency as f_c we then obtain the following specifications

- $f_c = 0.45/2$ End of passband frequency
- $f_s = 0.55/2$ Start of stopband frequency
- $|H(f)|^2 < 1/200^2, |f| > f_s$ Stopband attenuation

(c) The amplitude function of the Butterworth filter is monotonically decreasing. At $\Omega = \Omega_c$ it has an attenuation of -3 dB which thus should be associated with f_c. The next design point is thus the required attenuation at f_s as listed above which is the start of the stopband. We start by moving the specification to the continuous time. We only need to consider the frequencies since the amplitude function is invariant under the bilinear transformation.

- $\Omega_c = 2\tan(2\pi f_c/2) = 1.707$
- $\Omega_s = 2\tan(2\pi f_p/2) = 2.34$

To find the minimal Butterworth design we thus need to solve for N in the following equation

$$200^2 = 1 + (\Omega_s/\Omega_c)^{2N} \tag{14}$$

which yields

$$N = \frac{\ln(200^2 - 1)}{2\ln(2.34/1.707)} = 16.8 \tag{15}$$

Hence order 17 is needed to meet the specifications. Let $H(s)$ be the designed continuous time Butterworth filter of order 17. The corresponding digital filter is then obtained by changing s for $2\frac{z-1}{z+1}$. This yields a digital IIR filter of the same order. $\square$

5. Consider a filter in a state-space form where $x(n)$ is the input and $y(n)$ is the output

$$\begin{aligned} z(n+1) &= Az(n) + Bx(n) \\ y(n) &= Cz(n) + Dx(n). \end{aligned}$$

Classify the following special models as either IIR or FIR filters:

(a)

$$\begin{array}{cc} A = \begin{bmatrix} 0.8 & 2 \\ 0.1 & 0.8 \end{bmatrix} & B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ C = \begin{bmatrix} 1 & 1 \end{bmatrix} & D = 0 \end{array}$$

(2pt)

(b)

$$\begin{array}{cc} A = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} & B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ C = \begin{bmatrix} 1 & 1 \end{bmatrix} & D = 1 \end{array}$$

(2pt)

(c)

$$\begin{array}{cc} A = \begin{bmatrix} 0 & 0 \\ 0.1 & 0 \end{bmatrix} & B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ C = \begin{bmatrix} 1 & 1 \end{bmatrix} & D = 0 \end{array}$$

(2pt)

Solution: An FIR filter has a finite length impulse response. If the input is an impulse ($x(n) = \delta(n)$) then $y(n) = h(n) = CA^nB$ for $n > 0$. For (b) and (c) we directly note that $A^2 = 0$ which implies that $h(n) = 0$ for $k > 1$. Hence both filters are of FIR type. To analyze (a) take the Z-transform of the state-space equations and eliminate $Z(z)$. This yields

$$\begin{aligned} H(z) &= C(zI - A)^{-1}B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} z-0.8 & -2 \\ -0.1 & z-0.8 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \frac{2(z+0.25)}{(z-0.8)^2 - 0.2} = \frac{2(z+0.25)}{(z-\sqrt{2}-0.8)(z-\sqrt{2}+0.8)} \end{aligned} \tag{16}$$

which clearly is an IIR filter. $\square$

6. The optimal Wiener filter $h(k)$ minimizes the variance of $e(n)$ where

$$e(n) = d(n) - \sum_{k=0}^{N-1} h(k)x(n-k).$$

The filter input $x(n)$ and desired signal $d(n)$ are both zero mean and have auto- and cross-correlation functions

$$\gamma_{xx}(n) = \begin{cases} \sigma_x^2, & n = 0 \\ 0, & n \neq 0 \end{cases} \quad \gamma_{dx}(n) \begin{cases} \alpha_1, & n = 0 \\ \alpha_2, & n = 1 \\ 0, & n > 1, n < 0 \end{cases}$$

- What is the length of the optimal filter? (2pt)
- What are the optimal filter coefficients? (4pt)

Solution: The optimal FIR Wiener filter is obtained by

$$\mathbf{h} = \Phi_{xx}^{-1}\Gamma_{dx} \tag{17}$$

where the element i, j in Φ_{xx} is $\gamma_{xx}(i-j)$, element j in column vector Γ_{dx} is $\gamma_{dx}(j-1)$ and element j in column vector $\mathbf{h}$ is $h(j-1)$. Since $\gamma_{xx}(n)$ is is zero except for $n = 0$, the matrix Φ_{xx} is diagonal. Also note that only the two first elements in Γ_{dx} are non-zero. Consequently all filter coefficients except for the two first ones will be zero irrespectively of the size of $N > 2$. Hence the optimal filter is of length 2. (b) Evaluating $\mathbf{h} = \mathbf{\Phi_{xx}^{-1}\Gamma_{dx}}$ yields $h(0) = \alpha_1/\sigma_x^2$ and $h(0) = \alpha_2/\sigma_x^2$ $\square$

END

Good Luck!