SSY125 Digital Communications

Department of Electrical Engineering

Exam Date: January 18, 2020, 14:00-18:00
Location: HA, HB, HC

Teaching Staff Alexandre Graell i Amat, Arni Alfredsson, Andreas Buchberger

Material Allowed material is

- Chalmers-approved calculator.
- L. Råde, B. Westergren. Beta, Mathematics Handbook, any edition.
- A dictionary.

Grading A correct, clear and well-motivated solution gives a maximum of 15 points.

An erroneous answer, unclear, incomplete or badly motivated solutions give point reductions down to a minimum of 0 points. Answers in any other language than English are ignored.

Solutions The solutions are made available as soon as possible on the course web page.

Review The grading review will be on February 4, 2020, 10:00-11:00, and on February 11, 2020, 13:00-14:00, in room 6435A (Heisenberg) in the EDIT-building.

Grades The final grade on the course will be decided by the project (maximum score 30), quizzes (maximum score 3), tutorial grade (maximum score 7), and final exam (maximum score 60). The sum of all scores will decide the grade according to the following table.

Total Score	0–39	40–59	60–79	≥ 80
Grade	Fail	3	4	5

PLEASE NOTE THAT THE PROBLEMS ARE NOT NECESSARILY ORDERED IN DIFFICULTY.

Good luck!

Problem 1 - Channel Capacity [15 points]

Part I

Consider a discrete memoryless source whose output U takes values on the alphabet $\mathcal{U} = \{u_1, u_2\}$ with probabilities $P_U(u_1) = 0.6$ and $P_U(u_2) = 0.4$.

1. [*1 pt*] What is the source entropy?
2. [*1 pt*] Apply the Huffman coding algorithm to this source.
3. [*0.5 pt*] What is the expected codeword length?
4. [*0.5 pt*] What is the efficiency of the code?
5. [*5 pt*] Repeat questions 2–4 assuming that the source code is applied over three consecutive symbols of this source.

Part II

Consider a channel whose input is a random variable X that takes values on $\mathcal{X} = \{x_1, x_2\}$. The channel output is a random variable Y that takes values on $\mathcal{Y} = \{y_1, y_2, y_3\}$. The channel is defined by the conditional distribution $P_{Y|X}(y|x)$, given by

$$\begin{aligned}
P_{Y|X}(y_1|x_1) &= 1 - \varepsilon \\
P_{Y|X}(y_2|x_1) &= \varepsilon \\
P_{Y|X}(y_3|x_1) &= 0 \\
P_{Y|X}(y_1|x_2) &= 0 \\
P_{Y|X}(y_2|x_2) &= \varepsilon \\
P_{Y|X}(y_3|x_2) &= 1 - \varepsilon
\end{aligned}$$

where $\varepsilon \in [0, 1]$. Given are also two distributions for the input symbols: Either $P_X(x_1) = 0.5$, $P_X(x_2) = 0.5$, or $P_X(x_1) = 0.7$, $P_X(x_2) = 0.3$.

1. [*4 pt*] Compare both input distributions in terms of the resulting mutual information for this channel.
2. [*3 pt*] Compute the capacity of this channel.

Problem 2 - Modulation and Detection [15 points]

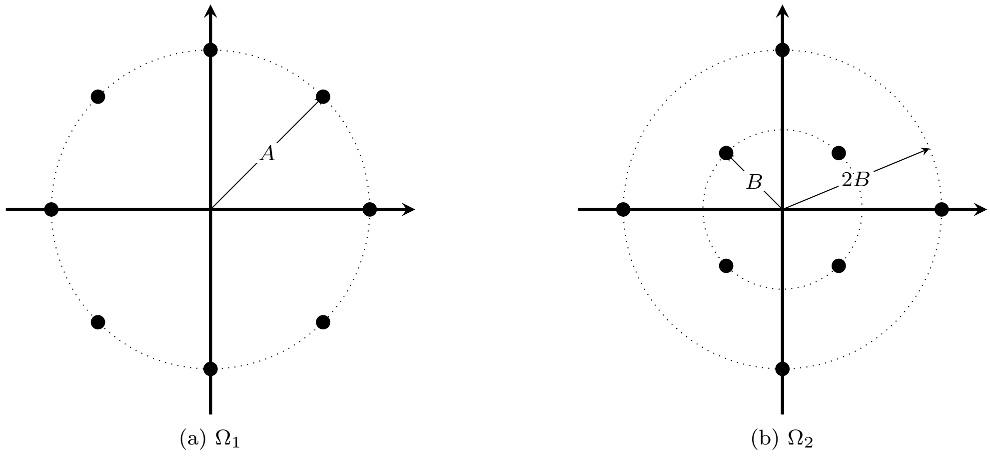

Figure 1: Constellations of two modulation formats.

Consider the two complex constellations Ω_1 and Ω_2 shown in Figures 1 (a) and (b). Assume that the received signal is given as $r = s + n$, where s is a point selected from one of the constellations and n is a realization of a zero-mean, complex Gaussian random variable N with $\mathbb{E}[|N|^2] = \mathsf{N}_0$. All constellation points in Ω_1 and Ω_2 are assumed equally likely to be transmitted.

Let $P_\mathsf{s}^{\Omega_1}$ and $P_\mathsf{s}^{\Omega_2}$ denote the symbol error probabilities for Ω_1 and Ω_2.

Questions

1. [*1 pt*] Determine A and B, such that both constellations Ω_1 and Ω_2 have average symbol energy E_s.
2. [*2 pt*] Find a Gray mapping for both constellations, if possible.
3. [*2 pt*] Carefully draw the maximum likelihood decision regions for both constellations.
4. [*3 pt*] Use the nearest neighbor approximation to compute $P_\mathsf{s}^{\Omega_2}$. The final expression should only be a function of $\mathsf{E}_\mathsf{s}/\mathsf{N}_0$. *Hint: The distance between two points on the unit circle with relative angle ϕ is* $2\sin(\phi/2)$. What happens to $P_\mathsf{s}^{\Omega_2}$ when $\mathsf{N}_0 \to 0$?
5. [*2 pt*] Assume $s \in \Omega_1$. Show that the maximum likelihood decision rule can be written as

$$\hat{s}_{\text{ML}} = \underset{s \in \Omega_1}{\operatorname{argmax}} \Re\{rs^*\},$$

   where s^* denotes the complex conjugate of s and $\Re\{\cdot\}$ denotes the real part of a complex number.

6. [*2 pt*] Suppose that the received signal is now given as $r = se^{j\theta} + n$, where θ is a random variable that is uniformly distributed between $[-\alpha, \alpha]$, for $\alpha > 0$. Assume that the symbols are detected by finding the constellation point that is closest to r in terms of Euclidean distance. What is the largest value that α can take such that $P_\mathsf{s}^{\Omega_1} \to 0$ when $\mathsf{N}_0 \to 0$? Justify your answer.
7. Suppose now that the received signal is given as $r = \beta s + n$, where β is a continuous random variable, uniformly distributed between $[0, 1]$.
   (a) [*1 pt*] Compute $\Pr(\beta = 0)$.
   (b) [*2 pt*] Assume that the symbols are detected by finding the constellation point that is closest to r in terms of Euclidean distance. Does $P_\mathsf{s}^{\Omega_1}$ or $P_\mathsf{s}^{\Omega_2}$ tend to zero when $\mathsf{N}_0 \to 0$? Justify your answer.

Problem 3 - Linear Block Codes and LDPC Codes [15 points]

Part I [4 points]

Consider a code $\mathcal{C}_1$ defined by the Tanner graph in Fig. 2.

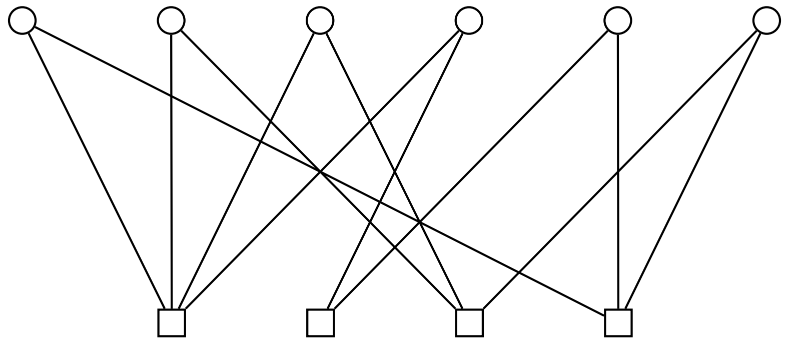

Figure 2: Tanner graph for $\mathcal{C}_1$.

1. [*1 pt*] Find the parity-check matrix $\boldsymbol{H}_1$ of $\mathcal{C}_1$.
2. [*1 pt*] Find the girth and highlight it in the graph.
3. [*2 pt*] Determine the variable node degree distribution $\Lambda(x)$ and the check node degree distribution $P(x)$. Is the LDPC code regular or irregular? Justify your answer!

Note: Even though $\mathcal{C}_1$ has small code length and is not sparse, consider it an LDPC code.

Part II [11 points]

Consider the code $\mathcal{C}_2$,

$$\mathcal{C}_2 = \left\{ \begin{matrix} (0 & 0 & 0 & 0 & 0 & 0), \\ (0 & 1 & 1 & 1 & 0 & 0), \\ (1 & 0 & 0 & 0 & 1 & 0), \\ (1 & 1 & 1 & 1 & 1 & 0), \\ (1 & 1 & 1 & 0 & 0 & 1), \\ (1 & 0 & 0 & 1 & 0 & 1), \\ (0 & 1 & 1 & 0 & 1 & 1), \\ (0 & 0 & 0 & 1 & 1 & 1) \end{matrix} \right\}.$$

1. [*2 pt*] Find the systematic parity-check matrix $\boldsymbol{H}_\mathsf{s}$ and systematic generator matrix $\boldsymbol{G}_\mathsf{s}$.
2. [*1 pt*] What are the code parameters (N, K, d_{min})? What is the code rate?
3. [*4 pt*] Assuming transmission over the binary symmetric channel with crossover probability $\epsilon = 0.3$, generate the complete syndrome table for $\boldsymbol{H}_\mathsf{s}$.
4. [*2 pt*] Suppose that you receive $\bar{\boldsymbol{y}} = (110000)$. What is the ML codeword based on the syndrome table?
5. [*2 pt*] Determine the dual code $\mathcal{C}_\perp$ of the code $\mathcal{C}_2$.

Problem 4 - Convolutional Codes and the Viterbi Algorithm [15 points]

Part I [11 points]

Consider the encoder $\mathcal{E}_1$ with block diagram shown in Fig. 3.

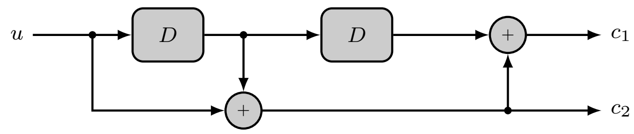

Figure 3: Encoder $\mathcal{E}_1$

1. [*1 pt*] Find the generator matrix for $\mathcal{E}_1$.
2. [*2 pt*] Transform $\mathcal{E}_1$ into recursive, systematic form $\mathcal{E}_{\mathsf{RSC}}$. Show the corresponding block diagram and generator matrix.
3. [*3 pt*] Draw one full section of the Trellis diagram of $\mathcal{E}_1$. Only display possible transitions and reachable states. Ensure that all state transitions are clearly labeled with the corresponding input and output bits.
4. [*5 pt*] Assume that the encoder $\mathcal{E}_1$ is initialized to the all-zero state and zero-termination. The bits are transmitted over an AWGN channel using BPSK where bit 0 is mapped to 1 and bit 1 is mapped to -1. The $\mathsf{E_b}/\mathsf{N_0} = 4$ dB and the received observation is given by

$$\boldsymbol{y} = \begin{pmatrix} 0.75 & -0.08 & 0.61 & -0.30 & -0.52 & -1.48 & 0.31 & -0.45 & 1.54 & 1.00 \end{pmatrix}.$$

   Find the maximum likelihood estimate of the information bits by using the Viterbi algorithm and hard decision decoding.

Part II [4 points]

Consider a code specified by the terminated trellis diagram in Fig. 4.

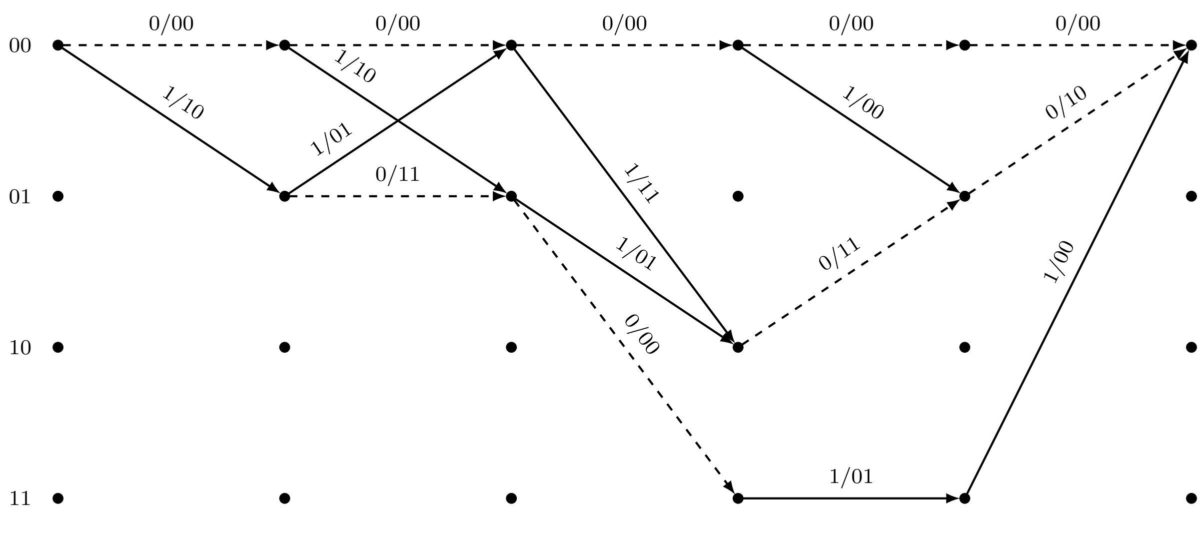

Figure 4: Trellis diagram.

1. [*2 pt*] Find all codewords.
2. [*1 pt*] Is it a linear code? Justify your answer.
3. [*1 pt*] What is the minimum distance of this code? Justify your answer.

For all parts of this problem, it is important that you clearly show all involved branch metrics, state metrics, and survivor paths.