Exam (January 18, 2020) Solution

Last modified January 30, 2020

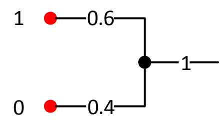
Problem 1 - Channel Capacity [15 points]

Part I

1. $\mathsf{H}(U) = \sum_i p_i \log_2(1/p_i) = 0.971$ bits
2. One possibility is

symbol	codeword	probability
u_1	1	0.6
u_2	0	0.4

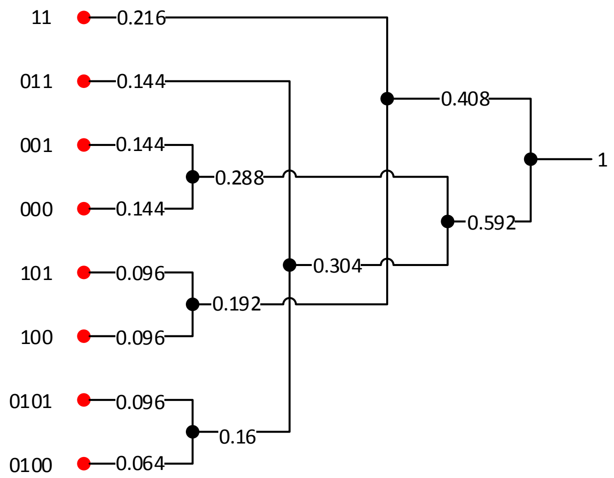
where the corresponding tree is



3. $\bar{L}_1 = 0.6 \cdot 1 + 0.4 \cdot 1 = 1$ bit
4. $\eta = \mathsf{H}(U)/\bar{L}_1 = 97.1\%$
5. Coding over three consecutive symbols yields a total of $2^3 = 8$ possible combinations:

symbol	codeword	probability
$u_1u_1u_1$	11	0.216
$u_1u_1u_2$	011	0.144
$u_1u_2u_1$	001	0.144
$u_1u_2u_2$	101	0.096
$u_2u_1u_1$	000	0.144
$u_2u_1u_2$	100	0.096
$u_2u_2u_1$	0101	0.096
$u_2u_2u_2$	0100	0.064

The tree corresponding to this code is



$\bar{L}_3 = 2 \cdot 0.216 + 3 \cdot 0.144 \cdot 3 + 3 \cdot 0.096 \cdot 2 + 4 \cdot 0.096 + 4 \cdot 0.064 = 2.944$ bits

$\eta = 3\mathsf{H}(U)/\bar{L}_3 = 98.94\%$

Part II

1. First recall the definition of the binary entropy function:

$$\mathsf{H}_b(p) = -p \log_2 p - (1-p)\log_2(1-p). \tag{1}$$

Let $p = P_X(x_1)$. With this we can write

$P_{Y\|X}$	$Y = y_1$	$Y = y_2$	$Y = y_3$
$X = x_1$	$1-\varepsilon$	ε	0
$X = x_2$	0	ε	$1-\varepsilon$

$P_{X,Y}$	$Y = y_1$	$Y = y_2$	$Y = y_3$
$X = x_1$	$p(1-\varepsilon)$	$p\varepsilon$	0
$X = x_2$	0	$(1-p)\varepsilon$	$(1-p)(1-\varepsilon)$

$$P_Y(y) = \begin{cases} p(1-\varepsilon), & y = y_1 \\ \varepsilon, & y = y_2 \\ (1-p)(1-\varepsilon), & y = y_3 \end{cases}$$

$P_{X\|Y}$	$Y = y_1$	$Y = y_2$	$Y = y_3$
$X = x_1$	1	p	0
$X = x_2$	0	$(1-p)$	1

Using the above probabilities, the following quantities can be computed.

$$\begin{aligned} \mathsf{H}(X) &= \mathsf{H}_b(p) \\ \mathsf{H}(Y) &= (1-\varepsilon)\mathsf{H}_b(p) + \mathsf{H}_b(\varepsilon) \\ \mathsf{H}(X|Y) &= \varepsilon \mathsf{H}_b(p) \\ \mathsf{H}(Y|X) &= \mathsf{H}_b(\varepsilon) \\ \mathsf{H}(X,Y) &= \mathsf{H}_b(p) + \mathsf{H}_b(\varepsilon). \end{aligned}$$

Therefore,

$$\begin{aligned} \mathsf{I}(X;Y) &= \mathsf{H}(Y) - \mathsf{H}(Y|X) \\ &= \mathsf{H}(X) - \mathsf{H}(X|Y) \\ &= \mathsf{H}(X) + \mathsf{H}(Y) - \mathsf{H}(X,Y) \\ &= (1-\varepsilon)\mathsf{H}_b(p) \end{aligned}$$

For the first input distribution, $p = 0.5$ and $\mathsf{I}(X;Y) = 1 - \varepsilon$.

For the second input distribution, $p = 0.7$ and $\mathsf{I}(X;Y) \approx 0.88(1-\varepsilon)$.

2. The capacity is given by the mutual information, maximized over all possible input distributions, i.e.,

$$C = \max_{p(x)} \mathsf{I}(X;Y)$$

For the given channel, the input distribution can be parametrized by one parameter p and hence we have a one-dimensional optimization problem

$$C = \max_{0 \le p \le 1} (1-\varepsilon)\mathsf{H}_b(p) = (1-\varepsilon) \max_{0 \le p \le 1} \mathsf{H}_b(p) = 1 - \varepsilon$$

where the second step follows because $\mathsf{H}_b(p)$ is maximized (and equal to one) when $p = 0.5$.

Problem 2 - Signal Constellations and Maximum Likelihood [15 points]

1. For Ω_1, the average energy per symbol is $\mathsf{E_s} = 8A^2/8 = A^2$, and therefore $A = \sqrt{\mathsf{E_s}}$.

   For Ω_2, the average energy per symbol is $\mathsf{E_s} = (4B^2 + 4 \cdot 4B^2)/8 = 5B^2/2$, and therefore $B = \sqrt{2\mathsf{E_s}/5}$.

2. An example of a Gray mapping for each constellation is depicted below.

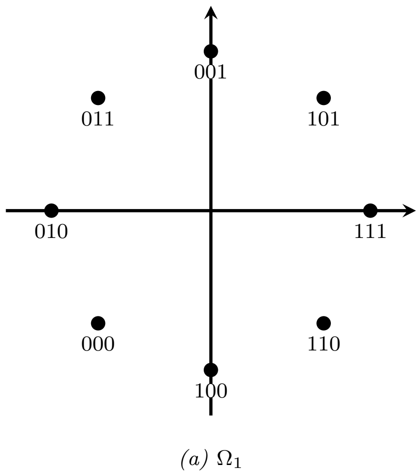


Figure 1: Gray mapping examples for the constellations.

3. The maximum likelihood decision regions can be seen below.

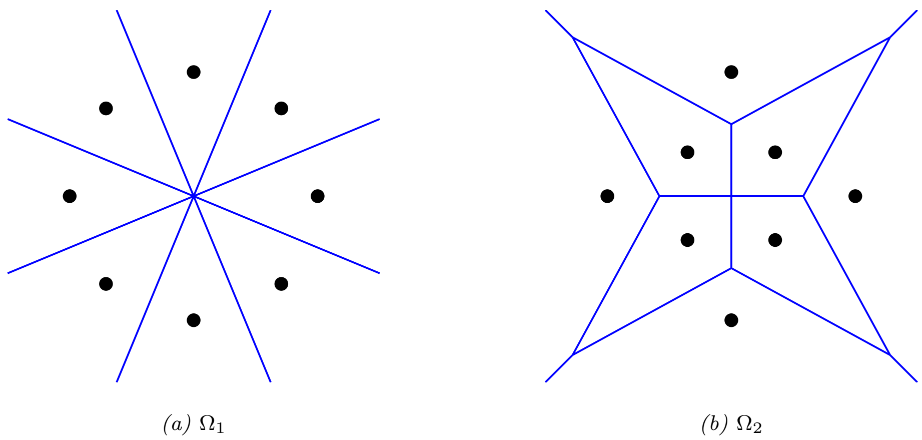


Figure 2: Maximum likelihood decision regions for the constellations.

4. Using the nearest neighbor approximation (see lecture notes for more details), $P_\mathsf{s}^{\Omega_2}$ is computed as

$$P_\mathsf{s}^{\Omega_2} \approx \bar{A}_{\min} \mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N_0}}}\right). \tag{2}$$

   By checking the Euclidean distance between all pairs in Ω_2, it can be verified that the minimum distance is $d_{\mathsf{E,min}} = 2B\sin(\pi/4) = \sqrt{2}B = \sqrt{4\mathsf{E_s}/5}$. This is the distance between points on the circle with radius

B. The average number of neighbors at this distance is $\bar{A}_{\min} = (4 \cdot 2 + 4 \cdot 0)/8 = 1$, and thus,

$$P_{\mathsf{s}}^{\Omega_2} \approx \bar{A}_{\min} \mathsf{Q}\left(\sqrt{\frac{d^2_{\mathsf{E,min}}}{2\mathsf{N}_0}}\right) = \mathsf{Q}\left(\sqrt{\frac{2\mathsf{E}_\mathsf{s}}{5\mathsf{N}_0}}\right). \tag{3}$$

Since $\mathsf{Q}(\,\cdot\,)$ decreases with increasing arguments, $\lim_{\mathsf{N}_0 \to 0} P_{\mathsf{s}}^{\Omega_2} = 0$.

5. When $s \in \Omega_1$, the maximum likelihood decision rule can be written as

$$\begin{aligned} \hat{s}_{\mathrm{ML}} &= \underset{s \in \Omega_1}{\operatorname{argmax}}\, p(r|s) \\ &= \underset{s \in \Omega_1}{\operatorname{argmin}}\, |r - s|^2 \\ &= \underset{s \in \Omega_1}{\operatorname{argmin}}(|r|^2 - 2\Re\{rs^*\} + |s|^2) \\ &= \underset{s \in \Omega_1}{\operatorname{argmin}}\, -2\Re\{rs^*\} \\ &= \underset{s \in \Omega_1}{\operatorname{argmax}}\, \Re\{rs^*\}, \end{aligned}$$

where $|r|^2$ can be discarded since r is not a function of s, and $|s|^2$ can also be discarded since it is constant due to Ω_1 being an 8PSK format.

6. When $\mathsf{N}_0 \to 0$, the phase noise caused by θ becomes the only possible source of errors. Since the symbol angles in Ω_1 are spaced apart by $\pi/4$, α should be less than $\pi/8$ to ensure that $P_{\mathsf{s}}^{\Omega_1} \to 0$ when $\mathsf{N}_0 \to 0$.

7. a) Since β is a continuous uniform random variable, $\Pr(\beta = 0) = 0$.

   b) In the considered received-signal model, β acts as amplitude noise. Since Ω_1 is constant modulus, i.e. all the constellation points have the same amplitude, β will not impact the symbol detection. This can be seen by looking at the detection regions for Ω_1 in part 3. If $\beta = 0$ symbol errors would occur, but $\Pr(\beta = 0) = 0$ and hence $P_{\mathsf{s}}^{\Omega_1} \to 0$ when $\mathsf{N}_0 \to 0$.

   For Ω_2, however, it is clear from the decision regions in part 3 that β will cause problems, since received constellation points on the outer circle will move towards the decision regions of the inner-circle points. Therefore, $P_{\mathsf{s}}^{\Omega_2}$ does not tend to zero when $\mathsf{N}_0 \to 0$.

Problem 3 - Linear Block Codes and LDPC Codes [15 points]

Part I

1.

$$\boldsymbol{H}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

2. The girth is the length of the shortest cycle in the Tanner graph, which is 4 in this case. It is highlighted in Fig. 3.

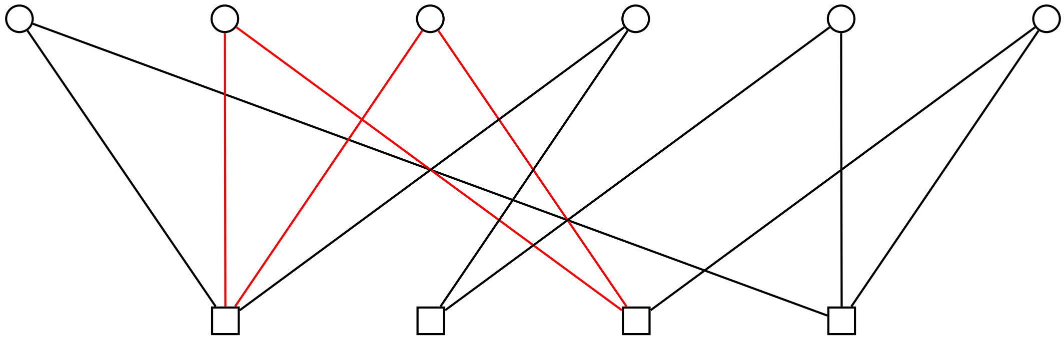


Figure 3: Tanner graph with highlighted girth.

3. By considering the Tanner graph, we get

$$\Lambda(x) = x^2 \qquad\qquad \mathsf{P}(x) = \frac{1}{4}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4.$$

Alternatively, we can get the same results from the parity check matrix $\boldsymbol{G}_1$ by considering the weight of the rows and the columns. The given code is an irregular LDPC code since the CNs are of different degrees.

Part II

1. The generator matrix has three rows. If we choose the second, the third, and the fifth codeword for the generator matrix, we get

$$\boldsymbol{G}_{\mathsf{s}} = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

From this, the parity-check matrix follow directly as

$$\boldsymbol{H}_{\mathsf{s}} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

2. The code parameters are $N = 6$, $K = 3$, $d_{\mathsf{min}} = 2$. The code rate follows as $R_{\mathsf{c}} = 3/6 = 1/2$.

3. The syndrome table can be found below.

syndrome	error vector
000	000000
001	001000
010	010000
011	000100
100	100000
101	101000
110	110000
111	000001

4. $\boldsymbol{s} = \bar{\boldsymbol{y}}\boldsymbol{H}_{\mathsf{s}}^T = (110)$. From the syndrome table, we note that this corresponds to the error patter $\boldsymbol{e} =$ 110000. Hence, $\hat{\boldsymbol{c}} = \bar{\boldsymbol{y}} + \boldsymbol{e} = (000000)$.

5. The parity-check matrix of $\mathcal{C}_2$, $\boldsymbol{H}_{\mathsf{s}}$, is the generator matrix of the dual code $\mathcal{C}_\perp$. Hence, we find the codewords of $\mathcal{C}_\perp$, $\tilde{\boldsymbol{c}}$ by calculating $\tilde{\boldsymbol{c}} = \boldsymbol{u}\boldsymbol{H}_{\mathsf{s}}$ for all $\boldsymbol{u} \in \{0, 1\}^3$. Hence,

$$\mathcal{C}_\perp = \left\{ \begin{matrix} (0 & 0 & 0 & 0 & 0 & 0), \\ (1 & 0 & 0 & 0 & 1 & 1) \\ (0 & 1 & 0 & 1 & 0 & 1) \\ (1 & 1 & 0 & 1 & 1 & 0) \\ (0 & 0 & 1 & 1 & 0 & 1) \\ (1 & 0 & 1 & 1 & 1 & 0) \\ (0 & 1 & 1 & 0 & 0 & 0) \\ (1 & 1 & 1 & 0 & 1 & 1) \end{matrix} \right\}.$$

Problem 4 - Convolutional Codes and the Viterbi Algorithm [15 points]

Part I

1.

$$\boldsymbol{G}_1 = \begin{pmatrix} 1 + D + D^2 & 1 + D \end{pmatrix}$$

2. We divide $\boldsymbol{G}$ by $1 + D + D^2$ and get

$$\boldsymbol{G}_{\mathsf{RSC}} = \begin{pmatrix} 1 & \frac{1+D}{1+D+D^2} \end{pmatrix}.$$

The corresponding block diagram is shown in Fig. 4.

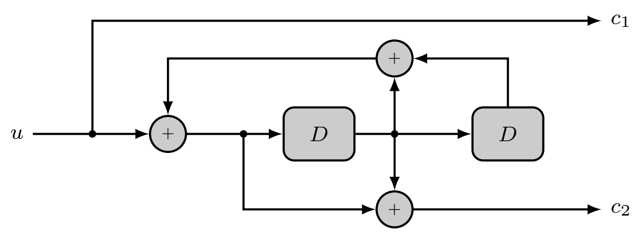


Figure 4: Encoder $\mathcal{E}_{\mathsf{RSC}}$

3. The trellis diagram is depcited in Fig. 5.

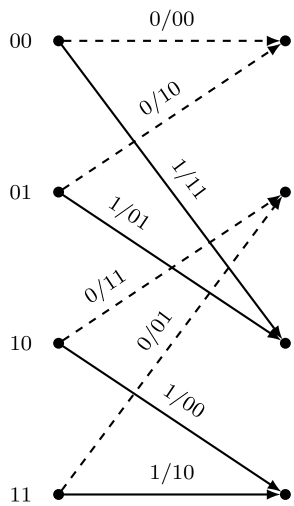


Figure 5: One section of the trellis.

4. In order to be able to do hard decision decoding, we require a binary received vector. From $\boldsymbol{y}$ and the mapping rule, we obtain

$$\bar{\boldsymbol{y}} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

We then use $\bar{\boldsymbol{y}}$ and the trellis from 3) to run the Viterbi algorithm as depicted in Fig. 6. We note that on three occasions, the cumulative weights are equal and we randomly discard a path. Hence, either choice is correct. Therefore, any of the codewords and correspond information bits

$$\boldsymbol{c}_1 = \begin{pmatrix}1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix} \qquad \boldsymbol{u}_1 = \begin{pmatrix}1 & 0 & 0 & 0 & 0\end{pmatrix}$$
$$\boldsymbol{c}_2 = \begin{pmatrix}0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\end{pmatrix} \qquad \boldsymbol{u}_2 = \begin{pmatrix}0 & 1 & 0 & 0 & 0\end{pmatrix}$$
$$\boldsymbol{c}_3 = \begin{pmatrix}0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0\end{pmatrix} \qquad \boldsymbol{u}_3 = \begin{pmatrix}0 & 0 & 1 & 0 & 0\end{pmatrix}$$
$$\boldsymbol{c}_4 = \begin{pmatrix}1 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0\end{pmatrix} \qquad \boldsymbol{u}_4 = \begin{pmatrix}1 & 1 & 1 & 0 & 0\end{pmatrix}$$

is correct.

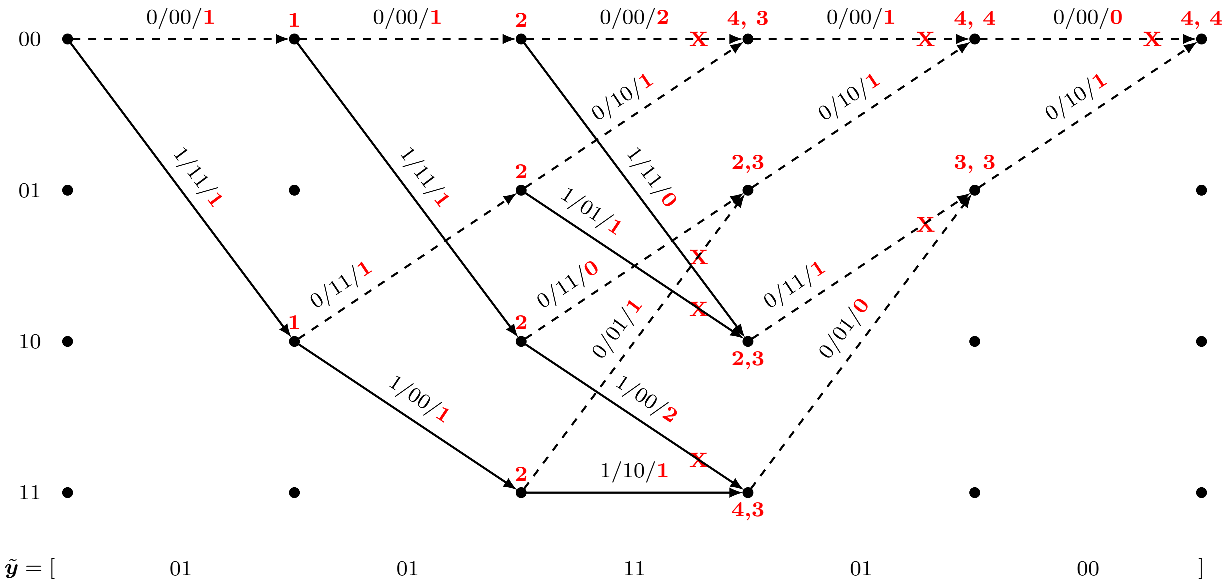


Figure 6: Viterbi algorithm.

Part II

1. The codewords correspond to paths in the trellis diagram. Hence we find all codewords by traversing every single path in the trellis. This leads to

$$\mathcal{C} = \left\{ \begin{array}{l}
\boldsymbol{c}_1 = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\\
\boldsymbol{c}_2 = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}\\
\boldsymbol{c}_3 = \begin{pmatrix}0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0\end{pmatrix}\\
\boldsymbol{c}_4 = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0\end{pmatrix}\\
\boldsymbol{c}_5 = \begin{pmatrix}0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\end{pmatrix}\\
\boldsymbol{c}_6 = \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}\\
\boldsymbol{c}_7 = \begin{pmatrix}1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0\end{pmatrix}\\
\boldsymbol{c}_8 = \begin{pmatrix}1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0\end{pmatrix}\\
\boldsymbol{c}_9 = \begin{pmatrix}1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0\end{pmatrix}\\
\boldsymbol{c}_{10} = \begin{pmatrix}1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0\end{pmatrix}
\end{array} \right\}.$$

2. For a code to be linear $\boldsymbol{c} + \tilde{\boldsymbol{c}} \in \mathcal{C}$ for all $\boldsymbol{c}, \tilde{\boldsymbol{c}} \in \mathcal{C}$ must hold. We note that

$$\boldsymbol{c}_2 + \boldsymbol{c}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \notin \mathcal{C}.$$

   Hence, the code is not linear.

3. We note that $\mathcal{C}$ does not contain any codeword multiple times and that $d_{\mathsf{H}}(\boldsymbol{c}_1, \boldsymbol{c}_2) = 1$. Hence, $d_{\mathsf{min}} = 1$.