MVE615/660, Flervariabelanalys I1/Z1

Lösningar tenta 1/6 – 22

1 (a) $\nabla f(x,y,z) = (e^x + y + z,\ x - 3y^2,\ x+1)$

$\Rightarrow \ n = \nabla f(0,1,2) = (1+1+2,\ 0-3\cdot 1^2,\ 0+1) = (4,-3,1)$

Planets ekv.: $n \cdot ((x,y,z) - (0,1,2)) = 0 \iff$

$\iff (4,-3,1)\cdot(x, y-1, z-2) = 0 \iff$

$\iff 4x - 3(y-1) + z - 2 = 0$

$\therefore \quad \underline{\underline{4x - 3y + z + 1 = 0}}$

(b) f avtar som snabbast i riktningen $-\nabla f$

$\Rightarrow$ Vill beräkna $\hat{u} = \dfrac{-\nabla f(0,1,2)}{|\nabla f(0,1,2)|} = \{(a)\} =$

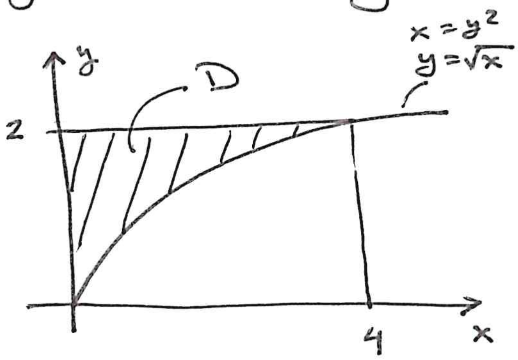
$= \dfrac{-(4,-3,1)}{\sqrt{4^2+(-3)^2+1^2}} = -\dfrac{(4,-3,1)}{\sqrt{26}}$

$D_{\hat{u}} f(0,1,2) = \hat{u}\cdot\nabla f(0,1,2) =$

$= -\dfrac{1}{\sqrt{26}}(4,-3,1)\cdot(4,-3,1) = -\dfrac{26}{\sqrt{26}} = \underline{\underline{-\sqrt{26}}}$

2 (a) Vill skriva $\int_0^4 \left(\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy \right) dx$ som en dubbelintegral och sedan byta integrationsordning.

Av integrationsgränserna framgår att vi har området D:



Detta ger:

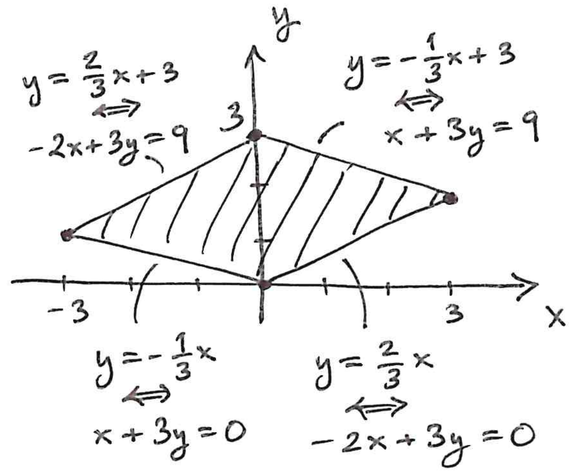
$$\int_0^4 \left(\int_{\sqrt{x}}^2 \frac{1}{y^3+1}\,dy \right) dx = \iint_D \frac{1}{y^3+1}\,dA =$$

$$= \int_0^2 \left(\int_0^{y^2} \frac{1}{y^3+1}\,dx \right) dy = \int_0^2 \frac{y^2}{y^3+1}\,dy =$$

$$= \left\{ \begin{array}{ll} t = y^3+1, & y=0 \leftrightarrow t=1 \\ dt = 3y^2\,dy, & y=2 \leftrightarrow t=9 \end{array} \right\} = \int_1^9 \frac{1}{t} \cdot \frac{dt}{3} =$$

$$= \frac{1}{3}\Big[\ln|t| \Big]_1^9 = \frac{1}{3}\ln(3^2) = \underline{\underline{\frac{2}{3}\ln(3)}}$$

(b) Av figuren framgår att vi ska göra variabelbytet:



$$\begin{cases} u = x + 3y & 0 \le u \le 9 \\ v = -2x + 3y & 0 \le v \le 9 \end{cases}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} 1 & 3 \\ -2 & 3 \end{vmatrix} = 3 + 6 = 9$$

$$\Rightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \frac{1}{\left| \frac{\partial(u,v)}{\partial(x,y)} \right|} = \frac{1}{|9|} = \frac{1}{9}$$

$$x = u - 3y \Rightarrow v = -2(u-3y) + 3y = -2u + 9y \Rightarrow$$

$$\Rightarrow y = \frac{1}{9}(2u+v)$$

$$\Rightarrow \iint_D y \sin(3y-2x)\,dx\,dy = \int_0^9 \left(\int_0^9 \frac{1}{9}(2u+v)\sin(v) \cdot \frac{1}{9}\,du \right) dv =$$

$$= \frac{1}{81} \int_0^9 \sin(v) \left[u^2 + uv \right]_{u=0}^{u=9} dv = \frac{1}{9} \int_0^9 (9+v)\sin(v)\,dv =$$

$$= \left[-\cos(v) \right]_0^9 + \frac{1}{9} \int_0^9 v \sin(v)\,dv = \{\text{partiell integration}\} =$$

$$= 1 - \cos(9) + \frac{1}{9}\left(\left[-v\cos(v) \right]_0^9 + \int_0^9 \cos(v)\,dv \right) =$$

$$= 1 - \cos(9) + \frac{1}{9}\left(-9\cos(9) + \sin(9) \right) =$$

$$= \underline{\underline{1 - 2\cos(9) + \frac{1}{9}\sin(9)}}$$

3 (a) $\frac{\partial f}{\partial x} = 2x e^{y-2x} - 2(x^2 - 2y)e^{y-2x} = (2x - 2x^2 + 4y)e^{y-2x}$

$\frac{\partial f}{\partial y} = -2e^{y-2x} + (x^2-2y)e^{y-2x} = (x^2 - 2y - 2)e^{y-2x}$

$$\nabla f = 0 \Rightarrow \begin{cases} x - x^2 + 2y = 0 & (*) \\ x^2 - 2y - 2 = 0 \Leftrightarrow 2y = x^2 - 2 \end{cases}$$

$2y = x^2 - 2$ insatt i $(*)$: $x - x^2 + x^2 - 2 = 0 \Leftrightarrow x = 2$

$x = 2$ insatt i $2y = x^2 - 2$: $2y = 4 - 2 \Leftrightarrow y = 1$

$\therefore$ $\underline{\underline{(2,1)}}$ kritisk punkt

$\frac{\partial^2 f}{\partial x^2} = (2-4x)e^{y-2x} - 2(2x - 2x^2 + 4y)e^{y-2x} = (2 - 8x + 4x^2 - 8y)e^{y-2x}$

$\frac{\partial^2 f}{\partial x \partial y} = 4e^{y-2x} + (2x - 2x^2 + 4y)e^{y-2x} = (4 + 2x - 2x^2 + 4y)e^{y-2x}$

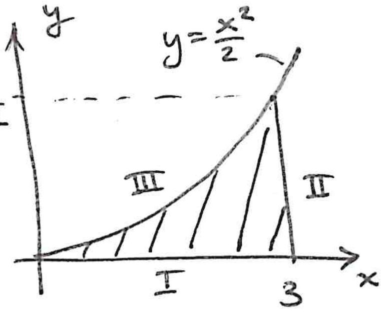
$\frac{\partial^2 f}{\partial y^2} = -2e^{y-2x} + (x^2 - 2y - 2)e^{y-2x} = (x^2 - 2y - 4)e^{y-2x}$

$$\Rightarrow \mathcal{H}(2,1) = \begin{pmatrix} -6e^{-3} & 4e^{-3} \\ 4e^{-3} & -2e^{-3} \end{pmatrix} \Rightarrow$$

$$\Rightarrow \det(\mathcal{H}(2,1)) = 12e^{-6} - 16e^{-6} < 0$$

$\therefore$ $\underline{\underline{(2,1) \text{ sadelpunkt}}}$

(b) Vet från (a) att f har den kritiska punkten $(2,1) \in D$ och att denna är en sadelpunkt, samt att den saknas singulära punkter.



Randpunkter:

I. $0 \le x \le 3,\ y=0$:

$$g(x) = f(x,0) = x^2 e^{-2x}$$

$$g'(x) = 2xe^{-2x} - 2x^2e^{-2x} = 2x(1-x)e^{-2x}$$

$$g'(x) = 0 \implies x_1 = 0,\ x_2 = 1 \implies$$

$$\implies (0,0) \in D \text{ ok!},\quad (1,0) \in D \text{ ok!}$$

II. $x=3,\ 0 \le y \le 9/2$:

$$g(y) = f(3,y) = (9-2y)e^{y-6}$$

$$g'(y) = -2e^{y-6} + (9-2y)e^{y-6} = (7-2y)e^{y-6}$$

$$g'(y) = 0 \implies 7-2y=0 \iff y = \frac{7}{2} \implies \left(3,\frac{7}{2}\right) \in D \text{ ok!}$$

III. $0 \le x \le 3,\ y = x^2/2$:

$$g(x) = f\left(x,\frac{x^2}{2}\right) = \left(x^2 - 2\cdot\frac{x^2}{2}\right)e^{\frac{x^2}{2}-2x} = 0$$

$$f(0,0) = 0$$

$$f(1,0) = (1-0)e^{0-2} = e^{-2}$$

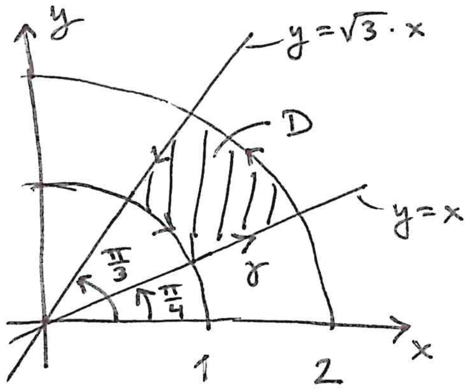
$$f\left(3,\frac{7}{2}\right) = \left(9-2\cdot\frac{7}{2}\right)e^{\frac{7}{2}-6} = 2e^{-\frac{5}{2}} = \frac{2}{\sqrt{e}}e^{-2} > e^{-2}$$

$\therefore$ Min.värde: 0, Max.värde: $2e^{-5/2}$

4. $\oint_\gamma \frac{1}{x}\arctan\left(\frac{y}{x}\right)dx - \arctan\left(\frac{y}{x}\right)dy = \oint_\gamma \mathbb{F}\cdot d\mathbf{r}$

där $\mathbb{F} = (F_1, F_2) = \left(\frac{1}{x}\arctan\left(\frac{y}{x}\right), -\arctan\left(\frac{y}{x}\right)\right)$

Vill använda Greens sats.



$$\frac{\partial F_2}{\partial x} = -\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{y}{x^2+y^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{1}{x}\cdot\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{1}{x^2+y^2}$$

$$\oint_\gamma \mathbb{F}\cdot d\mathbf{r} = \left\{\begin{matrix}\text{Greens}\\ \text{sats}\end{matrix}\right\} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)dxdy =$$

$$= \iint_D \frac{y-1}{x^2+y^2}\,dxdy = \left\{\begin{matrix}\text{Polära}\\ \text{koord.}\end{matrix}\right\} =$$

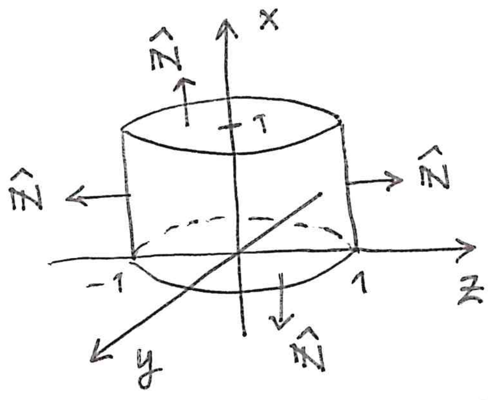
$$= \int_{\pi/4}^{\pi/3}\left(\int_1^2 \frac{r\sin\theta - 1}{r^2}\cdot r\,dr\right)d\theta = \int_{\pi/4}^{\pi/3}\left(\int_1^2\left(\sin\theta - \frac{1}{r}\right)dr\right)d\theta =$$

$$= \int_{\pi/4}^{\pi/3}\Big[r\sin\theta - \ln|r|\Big]_1^2\,d\theta = \int_{\pi/4}^{\pi/3}\left(\sin\theta - \ln(2)\right)d\theta =$$

$$= \Big[-\cos\theta - \theta\cdot\ln(2)\Big]_{\pi/4}^{\pi/3} = \Big\{$$  $$\Big\} =$$

$$= \left(-\frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{\pi}{3}\ln(2) + \frac{\pi}{4}\ln(2)\right) = \underline{\underline{\frac{\sqrt{2}-1}{2} - \frac{\pi}{12}\ln(2)}}$$

5. $$\oiint_{\partial K} \mathbb{F} \bullet \hat{\mathbb{N}}\, dS = \left\{ \begin{array}{l} \text{Gauss} \\ \text{sats} \end{array} \right\} =$$



(högerhandsregeln)

$$= \iiint_K \nabla \bullet \mathbb{F}\, dV = \iiint_K (1 + 6z^2)\, dV =$$

$$= \left\{ \begin{array}{ll} \text{Cylindriska koord.} & \\ x = x & , 0 \le x \le 1 \\ y = r\cos\theta & , 0 \le r \le 1 \\ z = r\sin\theta & , 0 \le \theta \le 2\pi \end{array} \right\} =$$

$$= \int_0^1 \left(\int_0^{2\pi} \left(\int_0^1 (1 + 6r^2 \sin^2\theta)\, r\, dr \right) d\theta \right) dx =$$

$$= \int_0^{2\pi} \left[\frac{1}{2} r^2 + \frac{6}{4} r^4 \sin^2\theta \right]_0^1 d\theta = \int_0^{2\pi} \left(\frac{1}{2} + \frac{3}{2} \sin^2\theta \right) d\theta =$$

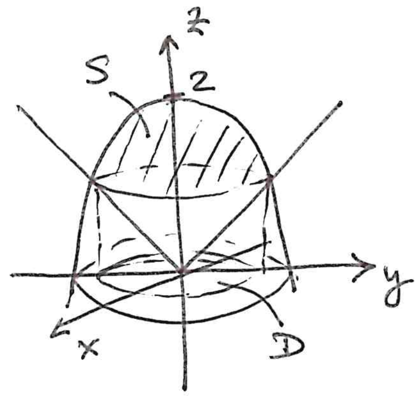
$$= \left\{ \begin{array}{l} \cos(2\theta) = 1 - 2\sin^2\theta \\ \Leftrightarrow \sin^2\theta = \dfrac{1 - \cos(2\theta)}{2} \end{array} \right\} = \int_0^{2\pi} \left(\frac{1}{2} + \frac{3}{2} \cdot \frac{1 - \cos(2\theta)}{2} \right) d\theta =$$

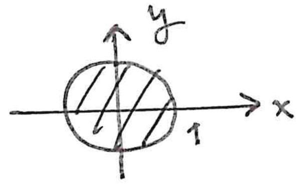
$$= \left[\frac{1}{2}\theta + \frac{3}{4}\left(\theta - \frac{\sin(2\theta)}{2} \right) \right]_0^{2\pi} = \pi + \frac{3\pi}{2} = \underline{\underline{\frac{5\pi}{2}}}$$

6 (a) Parametrisering av S:

$r(x,y) = (x, y, 2-x^2-y^2)$, $(x,y) \in D$

där $D \subset \mathbb{R}^2$ cirkelskivan med radie r som ges av skärningen mellan $z = 2-x^2-y^2$ och $z = \sqrt{x^2+y^2}$ i cylindriska koord.:





$$2 - r^2 = r \iff r^2 + r - 2 = 0$$

$$\Rightarrow r = -\frac{1}{2} \pm \sqrt{\frac{1}{4}+2} = -\frac{1}{2} \pm \frac{3}{2} \Rightarrow$$

$$\overset{r \geq 0}{\Rightarrow} \quad r = 1 \quad (\Rightarrow z = 1 \text{ för del (b)})$$

$$\frac{\partial r}{\partial x} = (1, 0, -2x), \quad \frac{\partial r}{\partial y} = (0, 1, -2y)$$

$$\Rightarrow \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ -2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$$

$$\text{Area}(S) = \iint_S 1 \, dS = \iint_D \left| \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} \right| dx\,dy =$$

$$= \iint_D \sqrt{1+4x^2+4y^2} \, dx\,dy = \left\{ \begin{array}{l} \text{Polära} \\ \text{koord.} \end{array} \right\} =$$

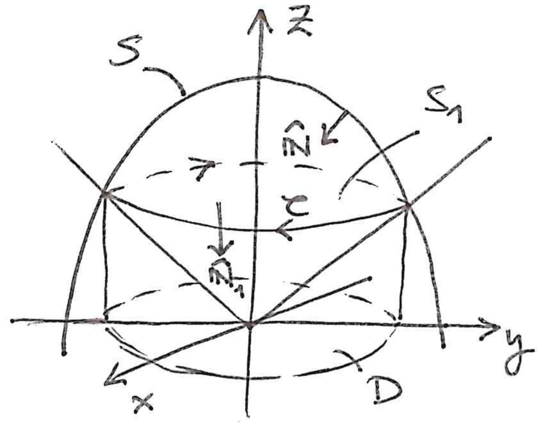
$$= \int_0^{2\pi} \left(\int_0^1 \sqrt{1+4r^2} \cdot r\,dr \right) d\theta = 2\pi \int_0^1 \sqrt{1+4r^2} \cdot r\,dr =$$

$$= \left\{ \begin{array}{ll} t = 1+4r^2, & r = 0 \leftrightarrow t = 1 \\ dt = 8r\,dr, & r = 1 \leftrightarrow t = 5 \end{array} \right\} = 2\pi \int_1^5 \sqrt{t} \cdot \frac{dt}{8} =$$

$$= \frac{\pi}{4} \left[\frac{t^{3/2}}{3/2} \right]_1^5 = \underline{\underline{\frac{\pi}{6}(5^{3/2} - 1)}} \text{ a.e.}$$

(b) $\nabla\times\mathbb{F} = \begin{pmatrix}\partial/\partial x\\ \partial/\partial y\\ \partial/\partial z\end{pmatrix} \times \begin{pmatrix} e^{x^2}-y \\ xz^2+\sin(y) \\ \cos(z^3)\end{pmatrix} =$

$= \left(-2xz \, , \, 0 \, , \, z^2+1\right)$



Parametrisering av S_1:

$\mathbb{r}(x,y) = (x, y, 1)$, $(x,y)\in D$ ← samma som i (a)

↑ följer av $r=1$ i (a)

$$\iint_S (\nabla\times\mathbb{F})\cdot\hat{\mathbb{N}}\, dS = \left\{\begin{matrix}\text{Stokes}\\ \text{sats}\end{matrix}\right\} = \oint_{\mathcal{C}} \mathbb{F}\cdot d\mathbb{r} =$$

↑ medurs

$$= \left\{\begin{matrix}\text{Stokes}\\ \text{sats}\end{matrix}\right\} = \iint_{S_1} (\nabla\times\mathbb{F})\cdot\hat{\mathbb{N}}_1\, dS =$$

$$= \iint_D \left((\nabla\times\mathbb{F})(x,y,1)\right)\cdot(0,0,-1)\,dx\,dy =$$

$$= \iint_D (-2x, 0, 2)\cdot(0,0,-1)\,dx\,dy =$$

$$= \iint_D -2\,dx\,dy = -2\cdot\text{Area}(D) = \underline{\underline{-2\pi}}$$

7. $\frac{\partial g}{\partial x} = \frac{\partial}{\partial x} f(x^2-y^2, 2xy) = f_1(x^2-y^2, 2xy)\cdot 2x + f_2(x^2-y^2, 2xy)\cdot 2y$

$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\cdot\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\left(2x\cdot f_1(x^2-y^2, 2xy) + 2y f_2(x^2-y^2, 2xy)\right) =$

$= 2\cdot f_1 + 2x\cdot\frac{\partial}{\partial x} f_1(x^2-y^2, 2xy) + 2y\cdot\frac{\partial}{\partial x} f_2(x^2-y^2, 2xy) =$

$= 2f_1 + 2x\left(f_{11}\cdot 2x + f_{21}\cdot 2y\right) + 2y\left(f_{12}\cdot 2x + f_{22}\cdot 2y\right) =$

$= \{f_{21} = f_{12}\} = 2f_1 + 4x^2 f_{11} + 8xy\cdot f_{12} + 4y^2\cdot f_{22}$

$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y} f(x^2-y^2, 2xy) = f_1(x^2-y^2, 2xy)\cdot(-2y) + f_2(x^2-y^2, 2xy)\cdot 2x$

$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\cdot\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}\left(-2y\cdot f_1(x^2-y^2, 2xy) + 2x\cdot f_2(x^2-y^2, 2xy)\right) =$

$= -2f_1 - 2y\left(f_{11}\cdot(-2y) + f_{21}\cdot 2x\right) + 2x\cdot\left(f_{12}\cdot(-2y) + f_{22}\cdot 2x\right) =$

$= \{f_{21} = f_{12}\} = -2f_1 + 4y^2 f_{11} - 8xy f_{12} + 4x^2 f_{22}$

$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = 2f_1 + 4x^2 f_{11} + 8xy f_{12} + 4y^2 f_{22} - 2f_1 +$

$+ 4y^2 f_{11} - 8xy f_{12} + 4x^2 f_{22} =$

$= (4x^2 + 4y^2) f_{11} + (4y^2 + 4x^2) f_{22} =$

$= 4(x^2+y^2)(f_{11} + f_{22}) = \{f \text{ harmonisk}\} = 0$ ∎