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MVE550 Stochastic Processes and Bayesian Inference

Re-exam August 22, 2022, 8:30 - 12:30
Examiner: Petter Mostad, phone 031-772-3579, visits exam at 9:30 and 11:30
Allowed aids: Chalmers-approved calculator
Total number of points: 30. At least 12 points are needed to pass.
See appendix for some information about some probability distributions

1. (6 points) Assume you have observed the values $x_1 = 2.3, x_2 = 5.1, x_3 = 7.9$ and you believe they are sampled from a Normal$(\mu, 1/\tau)$ distribution. You have some information about the parameters μ and τ; assume first that you know that $\mu = 4$ but are uncertain about τ.

   (a) Write down the likelihood function for τ. Find a probability density function of τ that is proportional to the likelihood.

   (b) You would like to compute the probability that your next observed value will be larger than 8. Describe in detail the steps in the Bayesian way of making such a computation. Describe and make the assumptions you need to make. You do not need to compute the actual probability, just describe how to compute it in mathematical detail and/or with R code.

   (c) Now assume that instead of knowing that $\mu = 4$, your prior information is a uniform distribution on the interval $[2, 6]$ for μ, and a uniform distribution on the interval $[0.1, 10]$ for τ. There are now several ways of (approximately) computing the posterior probability that $\mu > 4$; describe one of them.

2. (6 points)

   (a) Let $Z_0, Z_1, Z_2, \dots$ be a branching process, so that $Z_n = \sum_{i=1}^{Z_{n-1}} X_i$, where the X_i are independent copies of a random variable X. If

   $$\begin{aligned}\Pr[X = 0] &= 0.1\\ \Pr[X = 1] &= 0.5\\ \Pr[X = 3] &= 0.4\end{aligned}$$

   find the probability of extinction.

   (b) Assume instead that the offspring processes are different in even and odd generations, so that

$$
\begin{aligned}
Z_n &= \sum_{i=1}^{Z_{n-1}} X_i \qquad n \text{ even} \\
Z_n &= \sum_{i=1}^{Z_{n-1}} Y_i \qquad n \text{ odd}
\end{aligned}
$$

where the Y_i are independent copies of a random variable Y. Define

$$W = \sum_{i=1}^{Y} X_i.$$

Find and prove a relationship between the probability generating functions $G_X(s), G_Y(s), G_W(s)$ of X, Y, W, respectively.

(c) Assume $G_Y(s) = \exp(s-1)$. Describe the steps in a numerical way to compute the extinction probability for the process in part (b).

3. (6 points)

(a) For a counting process $\{N_t\}_{t\geq 0}$ to be a Poisson process with parameter λ, we must have $N_0 = 0$ and $N_t \sim \text{Poisson}(t\lambda)$ for all $t \geq 0$. Precisely describe two additional properties so that if N_t has these properties it must be a Poisson process.

(b) Using the definition above, write down a proof that if $\{N_t\}_{t\geq 0}$ is a Poisson process and T is the arrival time of the first event, then T has an exponential distribution.

(c) Assume that customers arrive at a carnival stand as a Poisson process with parameter λ. Each customer has a probability 0.01 of winning a grand price, a probability 0.1 of winning a smaller price, and a probability 0.89 of not winning. Find and simplify the formula for the following:

Given that no grand prices are won during the first hour of operation, and the first grand price is won before the end of the second hour, find the probability that the second grand price is also won before the end of the second hour.

4. (6 points) A continuous-time Markov chain has states 1,2,3,4,5. The expected holding times for these states are 3,2,1,1,2, respectively. The *embedded chain* has transition matrix

$$\tilde{P} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \end{bmatrix}.$$

Let $v = (3/7, 2/7, 1/14, 1/14, 1/7)$.

(a) Prove that v is a stationary distribution for the continuous-time Markov chain.

(b) Prove that v is *not* a limiting distribution for the continuous-time Markov chain.

(c) Find $\lim_{t\to\infty} \Pr[N_t = 3 \mid N_0 = 4]$.

5. (6 points) Each round in a game works as follows: You first pay 1 kroner. Then with a probability 1/10 you win 10 kroner and with a probability 9/10 you win nothing. Let X_i denote your total winnings or losses after i rounds.

   (a) Compute the expectation and variance of X_i.

   (b) Write $Y_i = aX_i$ for some $a > 0$, and find the a such that the variance of Y_i is i for all i.

   (c) What does the Donsker invariance principle say about the behaviour of Y_i when i is large?

   (d) Use the above to find an (approximate) value m so that with 95% probability the total winnings will never go above m during 10000 played rounds. (You may use that a variable with a standard normal distribution is in the interval $[-1.96, 1.96]$ with 95% probability).

Appendix: Some probability distributions

The Beta distribution

If $x \in [0, 1]$ has a Beta distribution with parameters with $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

We write $x \mid \alpha, \beta \sim \text{Beta}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Beta}(x; \alpha, \beta)$.

The Beta-Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Beta-Binomial distribution, with n a positive integer and parameters $\alpha > 0$ and $\beta > 0$, then the probability mass function is

$$\pi(x \mid n, \alpha, \beta) = \binom{n}{x} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}.$$

We write $x \mid n, \alpha, \beta \sim \text{Beta-Binomial}(n, \alpha, \beta)$ and $\pi(x \mid n, \alpha, \beta) = \text{Beta-Binomial}(x; n, \alpha, \beta)$.

The Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Binomial distribution, with n a positive integer and $0 \leq p \leq 1$, then the probability mass function is

$$\pi(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

We write $x \mid n, p \sim \text{Binomial}(n, p)$ and $\pi(x \mid n, p) = \text{Binomial}(x; n, p)$.

The Dirichlet distribution

If $x = (x_1, x_2, \ldots, x_n)$ has a Dirichlet distribution, with $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$ and with parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > 0, \ldots, \alpha_n > 0$, then the density function is

$$\pi(x \mid \alpha) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_n^{\alpha_n - 1}.$$

We write $x \mid \alpha \sim \text{Dirichlet}(\alpha)$ and $\pi(x \mid \alpha) = \text{Dirichlet}(x; \alpha)$.

The Exponential distribution

If $x \geq 0$ has an Exponential distribution with parameter $\lambda > 0$, then the density is

$$\pi(x \mid \lambda) = \lambda \exp(-\lambda x)$$

We write $x \mid \lambda \sim \text{Exponential}(\lambda)$ and $\pi(x \mid \lambda) = \text{Exponential}(x; \lambda)$. The expectation is $1/\lambda$ and the variance is $1/\lambda^2$.

The Gamma distribution

If $x > 0$ has a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x).$$

We write $x \mid \alpha, \beta \sim \text{Gamma}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Gamma}(x; \alpha, \beta)$.

The Geometric distribution

If $x \in \{1, 2, 3, \ldots\}$ has a Geometric distribution with parameter $p \in (0, 1)$, the probability mass function is

$$\pi(x \mid p) = p(1-p)^{x-1}$$

We write $x \mid p \sim \text{Geometric}(p)$ and $\pi(x \mid p) = \text{Geometric}(x; p)$. The expectation is $1/p$ and the variance $(1-p)/p^2$.

The Negative Binomial distribution

A stochastic variable x taking on as possible values any nonnegative integer has a Negative Binomial distribution if its probability mass function is given by

$$\pi(x \mid r, p) = \binom{x+r-1}{x} \cdot (1-p)^x p^r = \frac{\Gamma(x+r)}{\Gamma(x+1)\Gamma(r)} (1-p)^x p^r$$

where $r > 0$ and $p \in (0, 1)$ are parameters.

The Normal distribution

If the real x has a Normal distribution with parameters μ and σ^2, its density is given by

$$\pi(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

We write $x \mid \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2)$ and $\pi(x \mid \mu, \sigma^2) = \text{Normal}(x; \mu, \sigma^2)$.

The Poisson distribution

If $x \in \{0, 1, 2, \ldots\}$ has Poisson distribution with parameter $\lambda > 0$ then the probability mass function is

$$e^{-\lambda} \frac{\lambda^x}{x!}.$$

We write $x \mid \lambda \sim \text{Poisson}(\lambda)$ and $\pi(x \mid \lambda) = \text{Poisson}(x; \lambda)$. The Poisson distribution has expectation λ and variance λ.