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Suggested solutions to
MVE550 Stochastic Processes and Bayesian Inference

Re-exam August 22, 2022, 8:30 - 12:30

1. (a) We get

$$\begin{aligned}
\pi(\text{data} \mid \tau) &= \prod_{i=1}^{3} \text{Normal}(x_i; 4, 1/\tau) \\
&= \left(\frac{1}{\sqrt{2\pi/\tau}}\right)^3 \exp\left(-\frac{\tau}{2}\left((x_1-4)^2 + (x_2-4)^2 + (x_3-4)^2\right)\right) \\
&\propto_\tau \tau^{3/2} \exp\left(-\frac{\tau}{2}\left(1.7^2 + 1.1^2 + 3.9^2\right)\right) = \tau^{3/2}\exp(-9.655\tau).
\end{aligned}$$

This means that the likelihood $\pi(\text{data} \mid \tau)$ is proportional to the probability density $\text{Gamma}(\tau; 5/2, 9.655)$.

(b) To compute this probability you need to find the posterior predictive probability. To find this, you first need to find a posterior for τ, and this means you need to assume some prior. A choice corresponding with the computation in (a) is to choose a flat prior on the positive real values as a prior: With such a prior, the posterior becomes $\text{Gamma}(\tau; 5/2, 9.655)$.

The posterior predictive then becomes

$$\pi(x \mid \text{data}) = \int_0^\infty \text{Normal}(x; 4, 1/\tau)\,\text{Gamma}(\tau; 5/2, 9.655)\,d\tau$$

and the required probability can be computed as

$$\int_8^\infty \pi(x \mid \text{data})\,dx.$$

(One may provide more detail in several ways: One is to write

$$\begin{aligned}
&\int_8^\infty \int_0^\infty \text{Normal}(x; 4, 1/\tau)\,\text{Gamma}(\tau; 5/2, 9.655)\,d\tau\,dx \\
= &\int_0^\infty \left[\int_8^\infty \text{Normal}(x; 4, 1/\tau)\,d\tau\right] \text{Gamma}(\tau; 5/2, 9.655)\,dx
\end{aligned}$$

and note that this can be computed in R as a numerical integral of

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(1-pnorm(8,4,1/sqrt(tau)))*dgamma(tau, 5/2, 9.655).
```

Another is to compute

$$
\begin{aligned}
\pi(x \mid \text{data}) &= \int_0^\infty \frac{1}{\sqrt{2\pi/\tau}} \exp\left(-\frac{\tau}{2}(x-4)^2\right) \frac{9.655^{5/2}}{\Gamma(5/2)} \tau^{3/2} \exp(-9.655\tau)\, d\tau \\
&= \frac{9.655^{5/2}}{\sqrt{2\pi}\Gamma(5/2)} \int_0^\infty \tau^{3-1} \exp\left(-\tau(9.655 + (x-4)^2/2)\right) d\tau \\
&= \frac{9.655^{5/2}}{\sqrt{2\pi}\Gamma(5/2)} \cdot \frac{\Gamma(3)}{(9.655 + (x-4)^2/2)^3}.
\end{aligned}
$$

A third is to express this integral as a non-centered t-distribution.)

(c) A simple procedure is to use gridding: Make a uniform 2D grid for μ in the interval $[2, 6]$ and τ in the interval $[0.1, 10]$, for example with 100 grid points in each direction, for a total of 10000 grid points. Then compute the likelihood function

$$\pi(\text{data} \mid \mu, \tau) = \prod_{i=1}^{3} \text{Normal}(x_i; \mu, 1/\tau)$$

in each grid point, and normalize so that the values sum to 1. Then compute the sum at the grid points where $\mu > 4$.

2. (a) We get for the probability generating function

$$G_X(s) = 0.1 + 0.5s + 0.4s^2$$

and then

$$
\begin{aligned}
G_X(s) - s &= 0.1 \cdot (1 + 5s + 4s^2) - s \\
&= 0.1 \cdot (1 - 5s + 4s^2) \\
&= 0.1 \cdot (s-1)(4s^2 + 4s - 1) \\
&= 0.1 \cdot (s-1) \cdot 4 \cdot (s + 1/2 + \sqrt{2}/2)(s + 1/2 - \sqrt{2}/2)
\end{aligned}
$$

We see from this that the smallest positive root of $G_X(s) = s$, and thus the probability of extinction, is $\sqrt{2}/2 - 1/2$.

(b) We get

$$
\begin{aligned}
G_W(s) &= \text{E}(s^W) = \text{E}(\text{E}(s^W \mid Y)) = \text{E}\left(\text{E}\left(s^{\sum_{i=1}^{Y} X_i} \mid Y\right)\right) \\
&= \text{E}\left(\text{E}\left(\prod_{i=1}^{Y} s^{X_i} \mid Y)\right)\right) = \text{E}\left(\prod_{i=1}^{Y} \text{E}(s^{X_i})\right) = \text{E}\left(G_X(s)^Y\right) = G_Y(G_X(s))
\end{aligned}
$$

(c) By considering two consecutive generations as one generation, we see that the branching process can be viewed as a standard branching process with offspring process given by W. We also have

$$G_W(s) = G_Y(G_X(s)) = G_Y(0.1 + 0.5s + 0.4s^3) = \exp(0.5s + 0.4s^3 - 0.9).$$

To find the smallest positive root of $G_W(s)$ we can apply for example the R function `uniroot` to

$$f(s) = \exp(0.5s + 0.4s^3 - 0.9) - s$$

on the interval $[0, 1]$.

3. (a)
   - Stationary increments: For all $s, t > 0$ $N_{t+s} - N_s$ has the same distribution as N_t.
   - Independent increments: For $0 \leq q < r \leq s < t$, $N_t - N_s$ and $N_r - N_q$ are independent.

   (b) For any $t > 0$ we have that

$$\Pr[T > t] = \Pr[N_t = 0] = e^{-t\lambda},$$

using the probability mass function for the Poisson. Thus

$$\Pr[T \leq t] = 1 - e^{-t\lambda}$$

and taking derivative we get for the probability density for T

$$\pi(T) = \lambda e^{-t\lambda}.$$

Comparing with the density for the exponential distribution, we get $T \sim \text{Exponential}(\lambda)$.

   (c) As the winning of grand prices is a Poisson process and such processes have stationary increments, we can ignore the first hour and start the Poisson process at the start of the second hour. The required probability is the probability of two or more grand prices during this hour divided by the probability of one or more grand prices during this hour. If $(N_t)_{t\geq 0}$ is the Poisson process for grand prices, this can be computed as

$$\frac{\Pr[N_1 \geq 2]}{\Pr[N_1 \geq 1]} = \frac{1 - \Pr[N_1 = 0] - \Pr[N_1 = 1]}{1 - \Pr[N_1 = 0]} = \frac{1 - e^{-0.01\lambda}(1 + 0.01\lambda)}{1 - e^{-0.01\lambda}},$$

4. (a) From the expected holding times we get that $(q_1, q_2, \ldots, q_5) = (1/3, 1/2, 1, 1, 1/2)$. Using $\tilde{P}$ we can now compute the generator matrix as

$$Q = \begin{bmatrix} -1/3 & 1/3 & 0 & 0 & 0 \\ 1/2 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1 & 1/2 \\ 0 & 0 & 1/4 & 1/4 & -1/2 \end{bmatrix}.$$

We then get

$$vQ = (-\frac{1}{3}\frac{3}{7}+\frac{1}{2}\frac{2}{7}, \frac{1}{3}\frac{3}{7}-\frac{1}{2}\frac{2}{7}, -\frac{1}{14}+\frac{1}{2}\frac{1}{14}+\frac{1}{4}\frac{1}{7}, \frac{1}{2}\frac{1}{14}-\frac{1}{14}+\frac{1}{4}\frac{1}{7}, \frac{1}{2}\frac{1}{14}+\frac{1}{2}\frac{1}{14}-\frac{1}{2}\frac{1}{7}) = 0$$

proving that v is a stationary distribution.

(b) v cannot be a limiting distribution, as this Markov chain has no limiting distribution. The reason is that it is reducible, it has the two closed communication classes $\{1, 2\}$ and $\{3, 4, 5\}$. The most direct proof that the chain does not have a limiting distribution is to observe that the state when $t \to \infty$ depends on the starting state: It cannot move out of the communication class it starts in.

(c) When $N_0 = 3$, we know that the chain starts in the second communication class. Restricting the Markov chain to this class, it has generator matrix

$$Q' = \begin{bmatrix} -1 & 1/2 & 1/2 \\ 1/ & -1 & 1/2 \\ 1/4 & 1/4 & -1/2 \end{bmatrix}.$$

We have seen above that $(1/14, 1/14, 1/7)Q' = 0$. Normalizing so that this vector is a probability vector, we get that $v' = 14/4 \cdot (1/14, 1/14, 1/7) = (1/4, 1/4, 1/2)$ is the unique limiting distribution for the restricted chain. Thus

$$\lim_{t\to\infty} \Pr[N_t = 3 \mid N_0 = 4] = 1/4.$$

5. (a) Let Z denote the outcome of a single round. Then

$$\mathrm{E}(X_i) = i\,\mathrm{E}(Z) = i\left(9 \cdot \frac{1}{10} - 1 \cdot \frac{9}{10}\right) = 0$$

and

$$\mathrm{Var}(X_i) = i\,\mathrm{Var}(Z) = i(\mathrm{E}(Z^2) - \mathrm{E}(Z)^2) = i\,\mathrm{E}(Z^2) = i\left(9^2 \cdot \frac{1}{10} + 1 \cdot \frac{9}{10}\right) = 9i.$$

(b) We get $\mathrm{Var}(Y_i) = a^2\,\mathrm{Var}(X_i) = a^2 9i$, so setting $a = 1/3$ will lead to $\mathrm{Var}(Y_i) = i$.

(c) When i is large, Y_i behaves approximately like Brownian motion B_t with $t = i$.

(d) The maximum value of a Brownian motion on the interval $[0, 10000]$ can be written M_{10000} where we know from theory that M_{10000} has the same distribution as $|B_{10000}|$. But B_{10000} is normally distributed with expectation 0 and variance 10000, i.e., standard deviation $\sqrt{10000} = 100$. Using the hint, we know that B_{10000} is in the interval $[-196, 196]$ with 95% probability, so that

$$\Pr[M_{10000} < 196] = \Pr[|B_{10000}| < 196] = 0.95.$$

Thus, approximately, the maximum value of Y_i is below 196 with 95% probability during 10000 played rounds, so that the maximum value of X_i is below $3 \cdot 196 = 588$ with 95% probability during 10000 played rounds.