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Suggested solutions for
MVE550 Stochastic Processes and Bayesian Inference
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1. (a) Assume that we use the prior $p \sim \text{Beta}(\alpha, \beta)$ so that

$$\pi(p) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} p^{\alpha-1}(1-p)^{\beta-1}.$$

To prove conjugacy we need to prove that the posterior is then also a Beta distribution. We get

$$\begin{aligned}
\pi(p \mid x) &\propto_p \pi(x \mid p)\pi(p) \\
&\propto_p \binom{x+r-1}{x} \cdot (1-p)^x p^r p^{\alpha-1}(1-p)^{\beta-1} \\
&\propto_p p^{\alpha+r-1}(1-p)^{\beta+x-1} \\
&\propto_p \text{Beta}(p; \alpha+r, \beta+x)
\end{aligned}$$

which shows that the posterior is a Beta distribution.

(b) Note that the Uniform distribution on $[0, 1]$ is the same as a $\text{Beta}(1, 1)$ distribution. So using $\alpha = 1$ and $\beta = 1$ and repeated Bayesian update of the parameter, we get the posterior

$$\pi(p \mid \text{data}) = \text{Beta}(1+4+4+4, 1+4+2+3) = \text{Beta}(13, 10).$$

2. (a) First of all, for the purposes of this question, we may change the Markov chain by removing states above 7 and making state 7 into an absorbing state: This is OK because once the chain reaches state 7 it will never again return to lower states.

Let P be the transition matrix of this simplified chain:

$$P = \begin{bmatrix}
0 & 0.2 & 0.8 & 0 & 0 & 0 & 0 \\
0.8 & 0 & 0.2 & 0 & 0 & 0 & 0 \\
0 & 0.8 & 0 & 0.2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.8 & 0.2 & 0 & 0 \\
0 & 0 & 0 & 0.8 & 0 & 0.2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

Then the required probability is $(P^9)_{14}(P^5)_{46}$. This can be computed in R with

```
matrixpower(P, 9)[1,4]*matrixpower(P, 5)[4,6]
```

provided `matrixpower` computes the power of a matrix. We might also write this out as a matrix product

$$v_1 PPPPPPPPP v_4^t v_4 PPPPP v_6^t$$

where

$$\begin{aligned} v_1 &= \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix} \\ v_4 &= \begin{bmatrix}0 & 0 & 0 & 1 & 0 & 0 & 0\end{bmatrix} \\ v_6 &= \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix}. \end{aligned}$$

(b) The communication classes are $\{1, 2, 3\}$, $\{4, 5\}$, $\{6\}$, $\{7\}$, $\{8\}$, $\{9\}$, ... They are all open, and all states are transient, as for any states there is a nonzero probability that one will never return to this state.

(c) Making state 6 absorbing, we get

$$Q = \begin{bmatrix} 0 & 0.2 & 0.8 & 0 & 0 \\ 0.8 & 0 & 0.2 & 0 & 0 \\ 0 & 0.8 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.8 & 0.2 \\ 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}$$

and the answer is given by the sum of the first row of the fundamental matrix $F = (I - Q)^{-1}$. In R we might write

```
sum(solve(diag(5)-Q)[1,])
```

A mathematical way to write this might be

$$[1 \quad 0 \quad 0 \quad 0 \quad 0](I - Q)^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

3. (a) If X is a random variable with the offspring distribution have that $\mu = \mathrm{E}[X] = 0 \cdot (1 - a) + 1 \cdot 0 + 2 \cdot a = 2a$. We compare this number with 1, and conclude that the process is critical if $a = 1/2$, supercritical if $a > 1/2$, and subcritical if $a < 1/2$.

(b) The expected size is

$$\mu^n = (2a)^n.$$

(c) We first find the probability generating function:

$$G(s) = (1-a)s^0 + 0s^1 + as^2 = 1 - a + as^2.$$

Then we know that the extinction probability is the smallest positive root of $G(s) = s$. We get (using that we know $s = 1$ is always a root)

$$\begin{aligned} 1 - a + as^2 &= s \\ as^2 - s + 1 - a &= 0 \\ (s-1)(as + a - 1) &= 0. \end{aligned}$$

As $as + a - 1 = 0$ solves to

$$s = \frac{1-a}{a} = \frac{1}{a} - 1,$$

we see that if $a \leq 1/2$ the extinction probability is 1, while if $a > 1/2$ the extinction probability is $1/a - 1$.

(d) Let X be the original number of offspring and Y be the offspring from the Poisson process. Then

$$\begin{aligned} G(s) &= \mathrm{E}\left[s^{X+Y}\right] = \mathrm{E}\left[s^X\right]\mathrm{E}\left[s^Y\right] \\ &= (1 - \frac{2}{3} + \frac{2}{3}s^2)\sum_{k=0}^{\infty} s^k e^{-2}\frac{2^k}{k!} \\ &= \left(\frac{1}{3} + \frac{2}{3}s^2\right)e^{-2}\sum_{k=0}^{\infty}\frac{(2s)^k}{k!} \\ &= \frac{1+2s^2}{3}e^2 e^{2s} = \frac{1}{3}(1+2s^2)e^{2(s-1)} \end{aligned}$$

We want to find the smallest positive root of $G(s) = s$, so we study

$$f(s) = \frac{1}{3}(1+2s^2)e^{2(s-1)} - s.$$

As $\mathrm{E}[X+Y] = \frac{4}{3} + 2 > 1$ and the process is supercritical we know $f(s)$ will have a root in the interval $(0,1)$. In R one may use for example the function `uniroot`, on the function $f(s)$, making sure to aviod the root $s = 1$.

4. (a) We can compute

$$\begin{aligned} \mathrm{E}[N_2N_4] &= \mathrm{E}[N_2(N_4 - N_2 + N_2)] \\ &= \mathrm{E}\left[N_2(N_4 - N_2) + N_2^2\right] \\ &= \mathrm{E}[N_2]\,\mathrm{E}[N_4 - N_2] + \mathrm{E}\left[N_2^2\right] \\ &= \mathrm{E}[N_2]\,\mathrm{E}[N_2] + \mathrm{Var}[N_2] + \mathrm{E}[N_2]^2 \end{aligned}$$

As $N_2 \sim \text{Poisson}(2\lambda)$ we have from the appendix that $\mathrm{E}[N_2] = 2\lambda$ and $\mathrm{Var}[N_2] = 2\lambda$, so the answer becomes

$$\mathrm{E}[N_2N_4] = (2\lambda)^2 + 2\lambda + (2\lambda)^2 = 8\lambda^2 + 2\lambda.$$

(b) As $X_2 \sim$ Exponential(λ) and independently $X_4 \sim$ Exponential(λ) we get $\mathrm{E}[X_2X_4] = \mathrm{E}[X_2]\,\mathrm{E}[X_4] = \frac{1}{\lambda^2}$.

(c) A uniformly selected arrival has a uniform distribution on $[0, 4]$. Its expected value is thus $4/2 = 2$.

5. (a) We get

$$Q = \begin{bmatrix} -25/3 & 4 & 2 & 4/3 & 1 \\ 1/4 & -1/4 & 0 & 0 & 0 \\ 1/2 & 0 & -1/2 & 0 & 0 \\ 3/4 & 0 & 0 & -3/4 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix}$$

(b) As the Markov chain is a finite irreducible continuous-time Markov chain it has a unique stationary distribution which is the limiting distribution. This implies that as $t \to \infty$ the state it is in becomes independent of the starting state.

(c) We solve the system

$$v \begin{bmatrix} 1 & 4 & 2 & 4/3 & 1 \\ 1 & -1/4 & 0 & 0 & 0 \\ 1 & 0 & -1/2 & 0 & 0 \\ 1 & 0 & 0 & -3/4 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

were we write $v = [v_0, v_1, v_2, v_3, v_4]$. We quickly get the equations

$$\begin{aligned} v_1 &= 16v_0 \\ v_2 &= 4v_0 \\ v_3 &= \frac{16}{9}v_0 \\ v_4 &= v_0 \end{aligned}$$

Together with the equation $v_0 + v_1 + v_2 + v_3 + v_4 = 1$ we get $v_0 = \frac{9}{214}$, so the answer is $\frac{9}{214}$.

(d) We see directly that the transition rate graph is a star in this case, i.e., a tree, so the Markov chain is necessarily time reversible.

6. (a) A process G_t is geometric Brownian motion if there are parameters G_0, μ, and σ so that

$$G_t = G_0 e^{t\mu + \sigma B_t}$$

where B_t is Brownian motion.

(b) We get

$$\begin{aligned}
E(G_t) &= E(G_0 e^{t\mu+\sigma B_t}) \\
&= G_0 e^{t\mu} E(e^{\sigma B_t}) \\
&= G_0 e^{t\mu} \int_{-\infty}^{\infty} e^{\sigma s} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t} s^2\right) ds \\
&= G_0 e^{t\mu} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2t}(s^2 - 2\sigma t s)\right) ds \\
&= G_0 e^{t\mu} \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2t}(s - \sigma t)^2 + t\frac{\sigma^2}{2}\right) ds \\
&= G_0 e^{t\mu} e^{t\sigma^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}(s - \sigma t)^2\right) ds \\
&= G_0 e^{t(\mu+\sigma^2/2)}
\end{aligned}$$