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Suggested solutions for
MVE550 Stochastic Processes and Bayesian Inference
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1. (a) One way to solve this is to argue using symmetry: We must have that the probability becomes 0.5. More computationally, we may compute the marginal with

$$P(x_1 = 1) = \int_0^1 P(x_1 = 1 \mid p)\pi(p)\,dp = \int_0^1 p\,dp = \left[\frac{1}{2}p^2\right]_0^1 = \frac{1}{2}.$$

But it is also possible to look at this as Bayesian statistics, where p has the prior Beta$(1, 1)$ corresponding to the uniform distribution, and where $x_i \sim$ Binomial$(1, p)$. Using the Beta-Binomial conjugacy, we may compute using the prior predictive, which according to the Compendium becomes

$$P(x_1 = 1) = \text{Beta-Binomial}(1; 1, 1, 1) = \binom{1}{1}\frac{B(1+1, 1-1+1)}{B(1, 1)} = \frac{\Gamma(2)\Gamma(1)\Gamma(2)}{\Gamma(3)\Gamma(1)\Gamma(1)} = \frac{1}{2}$$

(b) Here it may be most easy to use the Beta-Binomial conjugacy. As the prior for p is Beta$(1, 1)$ and we update it with one observation of 1 and two of 0, the posterior becomes Beta$(2, 3)$.

(c) Here we may use the posterior predictive of the Beta-Binomial: We get

$$P(x_4 = 0 \mid \text{data}) = \text{Beta-Binomial}(0; 1, 2, 3) = \binom{1}{0}\frac{B(0+2, 1-0+3)}{B(2, 3)} = \frac{\Gamma(2)\Gamma(4)\Gamma(5)}{\Gamma(6)\Gamma(2)\Gamma(3)} = \frac{3}{5}.$$

An alternative is to use the find the marginal by integrating out the Beta$(2, 3)$ density derived in (b).

2. (a) If we write X_1 for the random variable representing the first set of children and X_2 for the possible extra child, we get

$$\begin{aligned} G(s) &= E(s^{X_1+X_2}) = E(s^{X_1}) \cdot E(s^{X_2}) = \left(\sum_{k=0}^{\infty} s^k \frac{2}{3^{k+1}}\right)(1 - a + sa) \\ &= \left(\frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{s}{3}\right)^k\right)(1 - a + sa) = \frac{2}{3} \cdot \frac{1}{1 - s/3}(1 - a + sa) = 2\frac{1 - a + sa}{3 - s}. \end{aligned}$$

(b) One way to compute is to use that the expected value of the offspring distribution is $G'(1)$: As $G'(s) = 2(2a + 1)/(3 - s)^2$ we get $G'(1) = a + 1/2$. There is a positive probability that the branching process will not go extinct if and only if this expectation is larger than 1, i.e., if $a > 1/2$. It is also easy to compute the expectation $a + 1/2$ directly from the model (not using $G(s)$).

(c) We find the probability of extinction by finding the smallest positive root of $G(s) = s$, in other words

$$2\frac{1 + a + sa}{3 - s} = s$$

or

$$s^2 + (2a - 3)s + 2 - 2a = 0$$

Using that one root is necessarily $s = 1$ we can factor this to get that the other root is $s = 2 - 2a$. When $1/2 < a < 1$ this is a number in the interval $(0, 1)$, so it is the smallest positive root.

(d) Given a, the probability of no offspring is $\frac{2}{3} \cdot (1 - a)$ as it corresponds to no offspring in both ways of producing children. Thus we get

$$\begin{aligned}
\pi(a \mid \text{data}) &\propto \pi(\text{data} \mid a)\pi(a) \\
&\propto \pi(\text{no offspring} \mid a)^r \pi(\text{some offspring} \mid a)^{N-r} \\
&\propto \left(1 - \frac{2}{3}(1 - a)\right)^r \left(\frac{2}{3}(1 - a)\right)^{N-r} \\
&\propto (1 + 2a)^r (1 - a)^{N-r}
\end{aligned}$$

3. (a) Particles arrive as a Poisson process with a total rate of $2.1 + 4.9 = 7$ per second. Thus, in an interval of length 0.5 the number of particles is distributed according to $\text{Poisson}(7 \cdot 0.5) = \text{Poisson}(3.5)$. The probability for 4 particles can be computed with

```
dpois(4, 3.5)
```

giving 0.1888123.

(b) The number of alpha particles among these 5 particles is distributed as $\text{Binomial}(5, \frac{2.1}{2.1+4.9}) = \text{Binomial}(5, 0.3)$. The probability for 3 alpha particles can be computed with

```
dbinom(3, 5, 0.3)
```

giving 0.1323.

(c) The arrival time of each of the 7 particles is uniform on the interval. Thus the probability that it comes in the first half of the interval is $1/2$. The probability that all particles come in the first half is $(1/2)^7 = 0.0078125$.

(d) What happens with the alpha particles is irrelevant. The sixth beta particle to arrive has an arrival time that is distributed as $\text{Gamma}(6, 4.9)$. The probability that it arrives in the interval given can be computed with

```
pgamma(2, 6, 4.9)-pgamma(1.7, 6, 4.9)
```

giving 0.08779957.

(e) The probability is $\frac{2.1}{2.1+4.9} = 0.3$.

4. (a) Using the notation of Dobrow, we get

$$Q = \begin{bmatrix} -0.5 & 0.3 & 0.1 & 0.1 \\ 0.2 & -0.25 & 0 & 0.05 \\ 0 & 0.5 & -0.8 & 0.3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and thus

$$V = \begin{bmatrix} -0.5 & 0.3 & 0.1 \\ 0.2 & -0.25 & 0 \\ 0 & 0.5 & -0.8 \end{bmatrix}.$$

We have that he fundamental matrix is $F = -V^{-1}$ and that the answer is the sum of the top row of F. In R we can compute as follows:

```
V <- matrix(c(-0.5, 0.3, 0.1, 0.2, -0.25, 0 , 0, 0.5, -0.8), 3, 3, byrow=T)
print(sum(-solve(V)[1,]))
```

giving 12.2619.

(b) We get

$$\tilde{P} = \begin{bmatrix} 0 & 0.6 & 0.2 & 0.2 \\ 0.8 & 0 & 0 & 0.2 \\ 0 & 0.625 & 0 & 0.325 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

5. (a) We have

$$P(8000 + 400 \cdot 2 + 500B_2 > 10000) = P(B_2 > 2.4)$$

As B_2 is normally distributed with expectation 0 and variance 2, this can be computed to be 0.0448, for example using the R command `1 - pnorm(2.4, 0, sqrt(2))`.

(b) We have that

$$8000 + 400 \cdot t + 500B_t > 10000$$

corresponds to $B_t > 4 - 0.8t$. If T is the smallest value such that $B_T = 4 - 0.8T$ then T is a stopping time. We may then compute (see Example 8.25 in Dobrow for details)

$$0 = E(B_0) = E(B_T) = 4 - 0.8E(T)$$

so solving for $E(T)$ gives $E(T) = 4/0.8 = 5$.