Petter Mostad
Applied Mathematics and Statistics
Chalmers and GU

MVE550 Stochastic Processes and Bayesian Inference

Exam January 9, 2021, 8:30 - 12:30
Examiner: Petter Mostad, phone 031-772-3579
Allowed aids: All aids are allowed.

For example you may access teaching material on any format and you may use R for computation. However, you are **not** allowed to communicate with any person other than the examiner and the exam guard. Total number of points: 30. To pass, at least 12 points are needed. You need to explain how you derive your answers, i.e., show the steps in computations, unless explicitly stated otherwise. There is an appendix containing relevant information about some probability distributions.

1. (4 points) Assume a variable $x > 0$ has density

$$\pi(x \mid \theta) = \frac{\theta^2 e^{-\theta/x}}{x^3}$$

where $\theta > 0$ is a parameter.

(a) Write down a proof that the Gamma family of densities is a conjugate family to the likelihood above.

(b) Assuming $\theta \sim \text{Gamma}(\alpha, \beta)$ and that $x \mid \theta$ has the distribution above, compute and simplify the marginal density for x.

2. (7 points) A Markov chain is defined as a random walk on the weighted undirected graph displayed in Figure 1. Note that the nodes are called A, B, C and the weights are $w_1, \dots, w_6$ where these are positive numbers.

(a) Given specific values for $w_1, \dots, w_6$, what is the limiting distribution for the Markov chain?

(b) Assume the chain has been observed for 28 steps, and that the table below lists counts of observed transitions from the node given on the left column to the node given on the top row.

	A	B	C
A	2	5	2
B	3	1	5
C	3	4	2

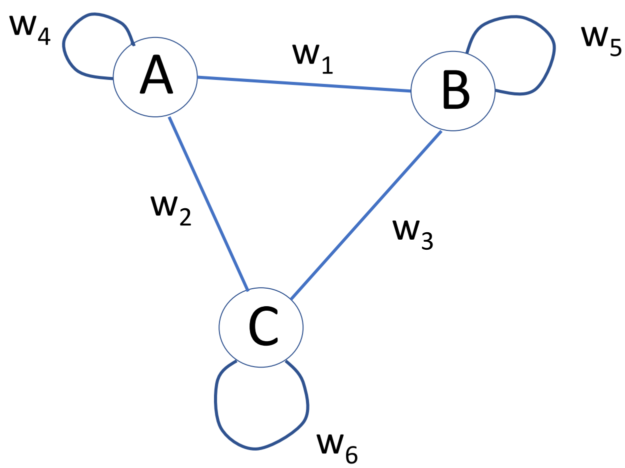

Figure 1: The graph for question 2.

Assume we use a prior for the weights with density $\pi(w_1, \ldots, w_6) = \exp(-w_1 - \cdots - w_6)$. Write down and simplify a function proportional to the posterior density for the weights $w_1, \ldots, w_6$.

(c) Describe in detail an algorithm that computes the (approximate) expected posterior limiting probability for the chain to be in state A. You may use R code or pseudo-code to give a precise description of your algorithm. You don't need to run the algorithm.

(d) In the situation above, we could have assumed that the Markov chain was represented by a stochastic matrix P and used Dirichlet priors for the rows of P. What, if any, would be the difference for the interpretation of the result? [1]

3. (6 points) Consider the Markov chain with states space $\{1, 2, \ldots, n\}$ and transition graph given in Figure 2, where p is a parameter satisfying $0 \leq p \leq 1$.

   (a) For each possible value of p determine the number of communication classes.

   (b) For each possible value of p and each state, determine its period.

   (c) For each possible value of p compute all possible stationary distributions for the chain, if any exist.

   (d) For each possible value of p compute all possible limiting distributions for the chain, if any exist.

[1] A better formulation of this question, unfortunately not used in the actual exam, would have been "What, if any, would be the difference in the posterior model if the amount of data approached infinity?"

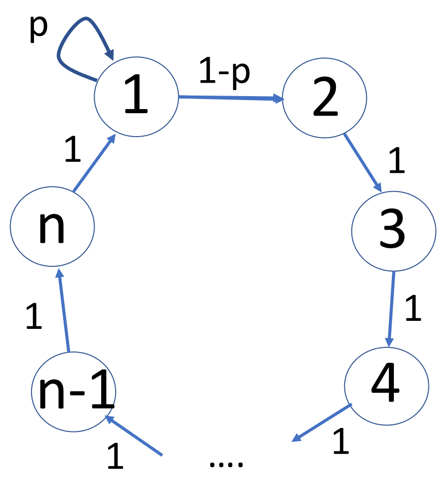

Figure 2: The transition graph for question 3

4. (4 points)

   (a) Assume the offspring process in a Branching process has probability 1/4 for zero offspring, probabiliy 1/2 for 1 offspring, and probability 1/4 for 3 offspring. Calculate the probability that the process will eventually go extinct.

   (b) Assume another Branching process uses as offspring process in the first generation a Poisson distribution with parameter λ. After this, the offspring distribution of (a) is used. Compute the probability that this branching process will go extinct.

5. (5 points) Adam is the main salesperson in a store that sells candy. Customers arrive according to independent Poisson processes. Adult customers arrive on average with one customer every 3 minutes while on average one child arrives every minute. The time it takes to service a customer is exponentially distributed. For adult customers it takes on average 2 minutes, while for child customers it takes on average 1 minute. If Adam is busy with a customer when another customer arrives, that customer moves on to another salesperson.

   (a) Compute the long time average proportion of time Adam serves adult customers.

   (b) Write down the transition rate graph for the process above, and also the graph with transition probabilities for a Poisson subordinated process to the process above.

   (c) Based on the above, give a short proof that the continuous-time Markov process you derived above is time reversible[2].

[2] A better formulation of the question would have been "give a proof that is as short as possible"

6. (2 points) Assume N_t is a Poisson process with parameter $\lambda = 2$. Prove that $N_t - 2t$ is a martingale with respect to N_t.

7. (2 points) Prove that the stochastic process $(X_t)_{0 \leq t \leq 1}$ defined by conditioning Brownian motion on $B_1 = a$ for some real a is identical to the process $Y_t = B_t - tB_1 + ta$ for $0 \leq t \leq 1$.

Appendix: Some probability distributions

The Bernoulli distribution

If $x \in \{0, 1\}$ has a Bernoulli distribution with parameter $0 \leq p \leq 1$, then the probability mass function is

$$\pi(x) = p^x(1-p)^{1-x}.$$

We write $x \mid p \sim \text{Bernoulli}(p)$ and $\pi(x \mid p) = \text{Bernoulli}(x; p)$.

The Beta distribution

If $x \in [0, 1]$ has a Beta distribution with parameters with $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

We write $x \mid \alpha, \beta \sim \text{Beta}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Beta}(x; \alpha, \beta)$.

The Beta-Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Beta-Binomial distribution, with n a positive integer and parameters $\alpha > 0$ and $\beta > 0$, then the probability mass function is

$$\pi(x \mid n, \alpha, \beta) = \binom{n}{x} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}.$$

We write $x \mid n, \alpha, \beta \sim \text{Beta-Binomial}(n, \alpha, \beta)$ and $\pi(x \mid n, \alpha, \beta) = \text{Beta-Binomial}(x; n, \alpha, \beta)$.

The Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Binomial distribution, with n a positive integer and $0 \leq p \leq 1$, then the probability mass function is

$$\pi(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

We write $x \mid n, p \sim \text{Binomial}(n, p)$ and $\pi(x \mid n, p) = \text{Binomial}(x; n, p)$.

The Dirichlet distribution

If $x = (x_1, x_2, \ldots, x_n)$ has a Dirichlet distribution, with $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$ and with parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > 0, \ldots, \alpha_n > 0$, then the density function is

$$\pi(x \mid \alpha) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} p_1^{\alpha_1-1} p_2^{\alpha_2-1} \cdots p_n^{\alpha_n-1}.$$

We write $x \mid \alpha \sim \text{Dirichlet}(\alpha)$ and $\pi(x \mid \alpha) = \text{Dirichlet}(x; \alpha)$.

The Exponential distribution

If $x \geq 0$ has an Exponential distribution with parameter $\lambda > 0$, then the density is

$$\pi(x \mid \lambda) = \lambda \exp(-\lambda x)$$

We write $x \mid \lambda \sim \text{Exponential}(\lambda)$ and $\pi(x \mid \lambda) = \text{Exponential}(x; \lambda)$. The expectation is $1/\lambda$ and the variance is $1/\lambda^2$.

The Gamma distribution

If $x > 0$ has a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x).$$

We write $x \mid \alpha, \beta \sim \text{Gamma}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Gamma}(x; \alpha, \beta)$.

The Geometric distribution

If $x \in \{1, 2, 3, \dots\}$ has a Geometric distribution with parameter $p \in (0, 1)$, the probability mass function is

$$\pi(x \mid p) = p(1-p)^{x-1}$$

We write $x \mid p \sim \text{Geometric}(p)$ and $\pi(x \mid p) = \text{Geometric}(x; p)$. The expectation is $1/p$ and the variance $(1-p)/p^2$.

The Normal distribution

If the real x has a Normal distribution with parameters μ and σ^2, its density is given by

$$\pi(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

We write $x \mid \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2)$ and $\pi(x \mid \mu, \sigma^2) = \text{Normal}(x; \mu, \sigma^2)$.

The Poisson distribution

If $x \in \{0, 1, 2, \dots\}$ has Poisson distribution with parameter $\lambda > 0$ then the probability mass function is

$$e^{-\lambda} \frac{\lambda^x}{x!}.$$

We write $x \mid \lambda \sim \text{Poisson}(\lambda)$ and $\pi(x \mid \lambda) = \text{Poisson}(x; \lambda)$.