Petter Mostad
Applied Mathematics and Statistics
Chalmers and GU

**MVE550 Stochastic Processes and Bayesian Inference**

Exam January 18, 2020, 14:00 - 18:00
**Allowed aids**: Chalmers-approved calculator.
Total number of points: 30. To pass, at least 12 points are needed
There is an appendix containing relevant information about some probability distributions.

1. (4 points) Assume a random variable $X$ with values in the positive real numbers has density $\pi(x \mid \theta) = \theta^2 x \exp(-\theta x)$, where $\theta > 0$ is a parameter. Assume we use the improper prior $\pi(\theta) \propto_\theta 1/\theta$.

   (a) Find a function of $\theta$ proportional to the posterior for $\theta$ given an observation $x$.

   (b) Find the name and parameters of the posterior for $\theta$ given an observation $x$.

   (c) Find the name and parameters of the posterior for $\theta$ after having made all the observations $x = 2.1$, $x = 1.2$, and $x = 3.9$.

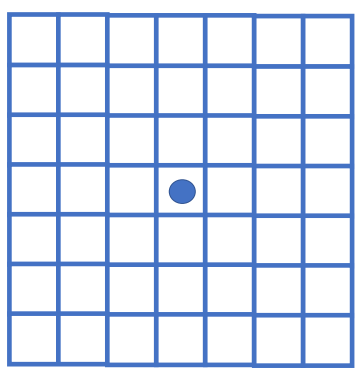

Figure 1: The board for question 2.

2. (4 points) A game is played on a board consisting of $7 \times 7$ squares, see Figure 1. The game starts in the square in the middle of the board. In each move, one moves with equal probability to one of the directly adjacent squares: There are 4, 3, or 2 such adjacent squares, with squares on the edges and in the corners having fewer neighbours.

(a) Show that the game can be described as a random walk on a graph.

(b) Compute the expected number of steps until you are back at the central square of the board.

3. (5 points) In the offspring process of a Branching process, the probability to obtain 3 offspring is 4/13 and the probability to obtain 2 offspring is also 4/13. With the remaining probability, there is no offspring.

(a) Is the process supercritical, subcritical, or critical?

(b) What is the expected size of $Z_n$, the size of the $n$'th generation?

(c) Find the probability generating function for the offspring distribution.

(d) Compute the probability that the process goes extinct.

4. (3 points) What is the Ising model? Describe how one can use Gibbs sampling on this model. (For partial credit: Just describe how Gibbs sampling works).

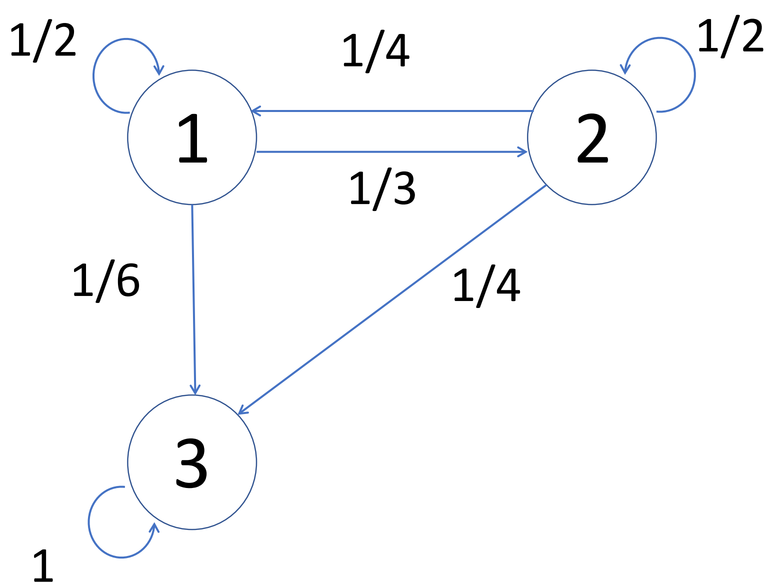


Figure 2: The network for question 5.

5. (4 points) Consider the discrete time Markov chain on the three states 1, 2, 3 illustrated in Figure 2. The chain starts in state 1.

(a) Describe the communication classes of the chain. For each state, describe whether it is transient or recurrent.

(b) Compute the expected number of visits to state 2.

6. (6 points) A biological system is described by a continuous time Markov chain with 5 diffeerent possible states: A, B, C, D, and E. We have that

   - If in state A, the system stays there for an expected time of 2 hours, then moves to state B.
   - If in state C, the system stays there for an expected time of 3 hours, then moves to state B.
   - If in state E, the system stays there for an expected time of 4 hours, then moves to state D.
   - If in state B, the system moves to state A with a rate of 0.5 per hour, to state C with a rate of 0.5 per hour, and to state D with a rate of 1 per hour.
   - If in state D, the system moves to state B at the rate of 1 per hour, and to state E with a rate of 0.5 per hour.

   (a) Write down the generator matrix $Q$.

   (b) Describe why this chain has a limiting distribution $v = (v_1, v_2, \dots, v_5)$.

   (c) Is this a time-reversible chain? Prove or disprove this.

   (d) Write down 5 equations whose joint solution determine $v$.

7. (4 points) Let $B_t$ denote Brownian motion, and define, for all $t > 0$ and some constants $a > 0$ and $b > 0$, $X_t = bB_{at}$.

   (a) Define what it means that $B_t$ is a Brownian motion.

   (b) Compute the distribution of $X_1 + X_2$, expressed in terms of $a$ and $b$.

   (c) Determine a formula $a$ and $b$ need to satisfy in order for $X_t$ to be Brownian motion, and prove that $X_t$ is Brownian motion if $a$ and $b$ satisfy this formula.

# Appendix: Some probability distributions

## The Bernoulli distribution

If $x \in \{0, 1\}$ has a Bernoulli distribution with parameter $0 \leq p \leq 1$, then the probability mass function is

$$\pi(x) = p^x(1-p)^{1-x}.$$

We write $x \mid p \sim \text{Bernoulli}(p)$ and $\pi(x \mid p) = \text{Bernoulli}(x; p)$.

## The Beta distribution

If $x \in [0, 1]$ has a Beta distribution with parameters with $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

We write $x \mid \alpha, \beta \sim \text{Beta}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Beta}(x; \alpha, \beta)$.

## The Beta-Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Beta-Binomial distribution, with $n$ a positive integer and parameters $\alpha > 0$ and $\beta > 0$, then the probability mass function is

$$\pi(x \mid n, \alpha, \beta) = \binom{n}{x} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(n+\alpha+\beta)}.$$

We write $x \mid n, \alpha, \beta \sim \text{Beta-Binomial}(n, \alpha, \beta)$ and $\pi(x \mid n, \alpha, \beta) = \text{Beta-Binomial}(x; n, \alpha, \beta)$.

## The Binomial distribution

If $x \in \{0, 1, 2, \ldots, n\}$ has a Binomial distribution, with $n$ a positive integer and $0 \leq p \leq 1$, then the probability mass function is

$$\pi(x \mid n, p) = \binom{n}{x} p^x (1-p)^{n-x}.$$

We write $x \mid n, p \sim \text{Binomial}(n, p)$ and $\pi(x \mid n, p) = \text{Binomial}(x; n, p)$.

## The Dirichlet distribution

If $x = (x_1, x_2, \ldots, x_n)$ has a Dirichlet distribution, with $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$ and with parameters $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_1 > 0, \ldots, \alpha_n > 0$, then the density function is

$$\pi(x \mid \alpha) = \frac{\Gamma(\alpha_1 + \alpha_2 + \cdots + \alpha_n)}{\Gamma(\alpha_1)\Gamma(\alpha_2)\cdots\Gamma(\alpha_n)} p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_n^{\alpha_n - 1}.$$

We write $x \mid \alpha \sim \text{Dirichlet}(\alpha)$ and $\pi(x \mid \alpha) = \text{Dirichlet}(x; \alpha)$.

## The Exponential distribution

If $x \geq 0$ has an Exponential distribution with parameter $\lambda > 0$, then the density is

$$\pi(x \mid \lambda) = \lambda \exp(-\lambda x)$$

We write $x \mid \lambda \sim \text{Exponential}(\lambda)$ and $\pi(x \mid \lambda) = \text{Exponential}(x; \lambda)$. The expectation is $1/\lambda$ and the variance is $1/\lambda^2$.

## The Gamma distribution

If $x > 0$ has a Gamma distribution with parameters $\alpha > 0$ and $\beta > 0$ then the density is

$$\pi(x \mid \alpha\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} \exp(-\beta x).$$

We write $x \mid \alpha, \beta \sim \text{Gamma}(\alpha, \beta)$ and $\pi(x \mid \alpha, \beta) = \text{Gamma}(x; \alpha, \beta)$.

## The Geometric distribution

If $x \in \{1, 2, 3, \ldots\}$ has a Geometric distribution with parameter $p \in (0, 1)$, the probability mass function is

$$\pi(x \mid p) = p(1-p)^{x-1}$$

We write $x \mid p \sim \text{Geometric}(p)$ and $\pi(x \mid p) = \text{Geometric}(x; p)$. The expectation is $1/p$ and the variance $(1-p)/p^2$.

## The Normal distribution

If the real $x$ has a Normal distribution with parameters $\mu$ and $\sigma^2$, its density is given by

$$\pi(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right).$$

We write $x \mid \mu, \sigma^2 \sim \text{Normal}(\mu, \sigma^2)$ and $\pi(x \mid \mu, \sigma^2) = \text{Normal}(x; \mu, \sigma^2)$.

## The Poisson distribution

If $x \in \{0, 1, 2, \ldots\}$ has Poisson distribution with parameter $\lambda > 0$ then the probability mass function is

$$e^{-\lambda} \frac{\lambda^x}{x!}.$$

We write $x \mid \lambda \sim \text{Poisson}(\lambda)$ and $\pi(x \mid \lambda) = \text{Poisson}(x; \lambda)$.