Chalmers tekniska högskola - Göteborgs universitet **Exam: 2021-01-04** $14^{00} - 18^{00}$.
MVE051/MVE055/MSG810 **Matematisk Statistik och Diskret Matematik**

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All aids permitted. The correct, well motivated solution gives the points given in parentheses for each task. Grade limits: Chalmers: 3: 12-17p, 4: 18-23p, 5: 24-30p, GU: G: 12-21, VG: 22-30.

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1. Consider a single throw of a six-sided die.

   Denote by A the event that the outcome is an even number, B that it is an odd number and $C = \{1, 2\}$ a third event.

   (a) What is the set Ω of possible outcomes? (1p)

   $\Omega = \{1, 2, 3, 4, 5, 6\}$.

   (b) What is the probability of $A \cap B$? (1p)

   $P(A \cap B) = 0$.

   (c) Are the events A and C independent? Justify your answer. (1p)

   Yes $P(A \cap C) = P(\{2\}) = 1/6$ and $P(A) \cdot P(C) = \frac{1}{2} \cdot \frac{2}{6} = \frac{1}{6}$.

2. A trebuchet[1] is a powerful siege engine that can launch a projectile such as a heavy stone over large distances using energy of a counterweight dropping from a height h. For a projectile thrown at an optimal angle, the throwing distance R is a function of the initial velocity v_0 of the projectile and is given by

$$R = \frac{v_0^2}{\mathrm{g}},$$

   with g the gravitational constant (neglecting drag and friction).

   By the law of preservation of energy, the initial velocity of the projectile depends on its mass X, on the mass M of the counterweight and the dropping height h of the counterweight via

$$\frac{X}{2} v_0^2 = M\mathrm{g}h.$$

   Assume the counterweight has mass $M = 150\,\mathrm{kg}$ and drops from a height of $2\,\mathrm{m}$. The projectiles have random mass X uniformly distributed between $10\,\mathrm{kg}$ and $12\,\mathrm{kg}$.

   Compute the expected value $\mathrm{E}[R]$ of the distance. Hint: Write R in terms of X and find the expectation of R using the density of X. (5p)

   First $R = 2\frac{M}{X}h$. Therefore we compute

$$E[R] = 2Mh \int_{10}^{12} \frac{1}{2x} \mathrm{d}x = Mh \log(6/5)\,\mathrm{kg}^{-1} \approx 54.7\,\mathrm{m}$$

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[1]Sv: Blida

3. In *Minecraft speedruns* players compete to complete the video game Minecraft as quickly as possible. A runner needs a certain number of *ender pearls* and *blaze rods* to access the end portals (a step in finishing the game.)

   Blaze rods are won by fighting blazes (a game creature). Upon being killed, each blaze has a 50% chance of dropping one blaze rod.

   To obtain ender pearls runners use a game mechanic known as bartering: In a barter, the player exchanges a gold ingot with a piglin (a game creature) for a randomly generated item. For each barter, there is a $\frac{20}{423} \approx 0.0473$ chance (in Mindcraft 1.16.1) the piglin will give the player ender pearls for the ingot.

   An analysis of speedruns by player "Dream" showed that 42 of the 262 piglin barters performed throughout these streams yielded ender pearls and 211 of the 305 killed blazes dropped blaze rods.

   Let X_{rods} be the random variable giving the number of blaze rods obtained after fighting 305 random blazes and X_{pearls} the number of times ender pearls were obtained in random 262 piglin barters. Assume independence of barters and blaze rod drops.

   The probability
   $$p = \mathrm{P}(X_{\text{pearls}} \geq 42 \text{ and } X_{\text{rods}} \geq 211)$$
   can be considered a measure of how unlikely "Dream"'s results in an unmodified game would be.

   (a) Determine the distribution of X_{rods} and its parameters. (1p)

   $X_{\text{rods}} \sim \mathrm{Bin}(305, \frac{1}{2})$

   (b) Determine a normal approximation to X_{rods}. (1p)

   $X_{\text{rods}} \overset{\text{approx}}{\sim} \mathrm{N}(152.5, 76.25)$

   (c) Assume $\mathrm{P}(X_{\text{pearls}} \geq 42) = 5.65 \cdot 10^{-12}$.
   Use the normal approximation to compute p (approximately). (2p)

   $\approx$ 5.65e-12 $\cdot$ 1.04e-11.

   (d) Formulate a null-hypothesis and an alternative hypothesis (in words) for which p is a proper p-value. What is your conclusion for the corresponding test at significance level $\alpha = 1\%$? (2p)

   H_0: Dream plays an unmodified game, H_1: Dream has modified the item generation to make droppings and or successful barters more likely (one-sided alternative).

4. Moore's law, named after Gordon Moore, director of research and development at Fairchild Semiconductor, states that the number of transistors in a dense integrated circuit (IC) doubles about every two years. The rapid miniaturisation of metal–oxide–semiconductor (MOS) transistors is a key driving force behind Moore's law.

   The table below gives the channel length C_i in [nm] used in MOS production for a number of years:

t_i	1987	1990	1993	1996	2005	2007	2016	2018
C_i	800	600	350	250	65	45	10	7

$$\sum_{i=1}^{8} t_i = 16012 \quad \sum_{i=1}^{8} \log_2 C_i = 52.933$$

$$\sum_{i=1}^{8} t_i^2 = 32048988 \quad \sum_{i=1}^{8} (\log_2 C_i)^2 = 398.398 \quad \sum_{i=1}^{8} t_i \log_2 C_i = 105730$$

Remark: Calculate with sufficient precision.

(a) Find the regression line for the logarithmic (basis 2) channel length $c_i = \log_2 C_i$ with the time t_i as explanatory variable. (2p)

$$c_i = 451.98 - 0.2225 t_i + \epsilon_i$$

(b) Give an interpretation of the slope: how many years does it take for the channel length to be reduced by a factor of 2? (1p)

About $4\frac{1}{2}$ years.

(c) Is there any significant linear relation between the channel length and time? Formulate a null hypothesis and test at significance level $\alpha = 0.01$. Assume normal errors with unknown variance σ^2. (2p)

$S^2 = 0.0215$. $t_c = 3.707$ is the $1 - \alpha/2$ quantile of the $t(8-2)$-distribution.

$$T = \frac{\beta_1}{S/\sqrt{\sum(x_i - \bar{x}_i)^2}} = -47.26$$

is more extreme than t_c, hence reject the null hypothesis that there is no linear relationship.

(d) Find a 95% confidence interval for the slope. Use it to determine a 95% confidence interval for the time until the channel length is reduced by a factor 2. (2p)

$CI_{slope} = [-0.234036, -0.210998]$. Therefore $CI_{years} = [4.2728, 4.7394]$.

5. A toaster and a coffee machine are connected via a data link to a computer and can transmit messages ("coffee is ready" respective "toast is ready") to the user. The data link can only be used by one machine at a time, if both machines send a message at the same time, *packet loss* occurs and no message is received at all. If only one machine attempts to send a message via the data link, the message transfer is successful. In order to resolve conflict, both machines attempt to send their message only with a certain probability in each round, and keep trying until they succeed.[2]

   The coffee machine attempts to send its message with probability $\frac{1}{2}$ and the toaster sends its message with probability $\frac{1}{3}$, independently.

   Assume coffee and a toast is ready.

   (a) What is the probability that no message is successfully transmitted in the first round. (1p)

   $(1 - \frac{1}{3})(1 - \frac{1}{2}) + \frac{1}{2}\frac{1}{3} = \frac{1}{2}$.

   (b) What is the probability that both machines successfully transmit their message to the user in the first two rounds. (1p)

   $\frac{1}{2}(1 - \frac{1}{3})\frac{1}{3} + (1 - \frac{1}{2})\frac{1}{3}\frac{1}{2} = \frac{1}{9} + \frac{1}{12}$.

[2] If one machine has successfully sent a message, it turns silent, while the other machine continues to randomly attempt to send its message in the next rounds.

(c) The procedure can be modelled by a Markov chain with four states:

$E_1 = \{$Both machines need to send a message$\}$

$E_2 = \{$Only the coffee machine needs to send a message$\}$

$E_3 = \{$Only the toaster needs to send a message$\}$

$E_4 = \{$No messages need to be send$\}$

Complete the following stochastic transition matrix with entries $p_{ij} = \mathrm{P}(E_j \mid E_i)$. (2p)

$$\mathbf{P} = \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} ? & ? & ? & 0 \\ 0 & ? & ? & ? \\ 0 & ? & ? & ? \\ ? & ? & ? & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}$$

$$\mathbf{P} = \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & E_3 & E_4 \end{array} \\ \begin{bmatrix} 1/2 & 1/6 & 2/6 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 2/3 & 1/3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ E_1 \\ E_2 \\ E_3 \\ E_4 \end{array}$$

(c) What are the absorbing states of a Markov chain with transition matrix $\mathbf{P}$? (1p)

E_4.

6. Use generating functions to find an explicit form for a_n where a_n is defined as follows (4p)

$$\begin{cases} a_0 = 0 \\ a_1 = 1 \\ a_2 = 4 \\ a_n = 2a_{n-1} + n \quad n > 0 \end{cases}$$

Hint: $\sum\limits_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2} = -\frac{1}{1-x} + \frac{1}{(1-x)^2}$ when $|x| < 1$.

We are looking for $a_0, \ldots, a_n, \ldots$ Let

$$f(x) = \sum_{k=0}^{\infty} a_k x^k.$$

Now

$$f(x) = a_0 + \sum_{k=1}^{\infty} (2a_{k-1} + k)x^k = 0 + 2x \sum_{k=0}^{\infty} a_k x^k + \sum_{k=1}^{\infty} kx^k$$
$$= 2xf(x) + \frac{x}{(1-x)^2} \quad |x| < 1$$

Thus

$$f(x) = \frac{1}{1-2x} \cdot \frac{x}{(1-x)^2} = ... = -\frac{2}{2x-1} - \frac{1}{1-x} - \frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} 2^{k+1} x^k - \sum_{k=0}^{\infty} x^k - \sum_{k=0}^{\infty} (1+k) x^k$$

thus we have identified the expansion of f and read its coefficients as

$$a_n = -n + 2^{n+1} - 2$$

Lycka till!