Lösning:

1. Let $P = \{\text{test is positive}\}$
$D = \{\text{the person has the disease}\}$

$P(P/D) = 0{,}95$ $\qquad$ $P(P/D^c) = 0{,}10$

$P(D) = 0{,}005$ $\qquad$ $P(D^c) = 1 - P(D) = 1 - 0{,}005 = 0{,}995$

a) $P(P) = P(P/D) \cdot p(D) + p(P/D') \cdot p(D')$

$= 0{,}95 \cdot 0{,}005 + 0{,}10 \cdot 0{,}995 = 0{,}10425$

b) $P(D/P) = \dfrac{P(P/D) \cdot P(D)}{P(P)}$ (Bayes' thm)

$= \dfrac{0{,}95 \cdot 0{,}005}{0{,}10425} = 0{,}0456$

c) $P(D^c/P^c) = \dfrac{P(P^c/D^c) \cdot p(D^c)}{p(P^c)} = \dfrac{(1 - P(P/D^c)) \cdot p(D^c)}{1 - p(P)}$

$= \dfrac{(1 - 0{,}10) \cdot 0{,}995}{1 - 0{,}10425} = 0{,}9997.$

2. $\sigma_x = \sigma_y = \sigma.$

$$\bar{x} = \frac{\sum x_i}{10} = \frac{421{,}4}{10} = 42{,}14 \qquad \bar{y} = \frac{\sum y_i}{10} = \frac{432{,}3}{10} = 43{,}23$$

$$s_x^2 = \frac{\sum (x_i - \bar{x})^2}{9} = \frac{4{,}204}{9} = 0{,}467 \qquad s_y^2 = \frac{\sum (y_i - \bar{y})^2}{9} = \frac{5{,}061}{9} = 0{,}562$$

Pooled variance: $s_p^2 = \dfrac{(n_1-1)s_x^2 + (n_2-1)s_y^2}{n_1+n_2-2} = \dfrac{9 \cdot 0{,}467 + 9 \cdot 0{,}562}{18}$

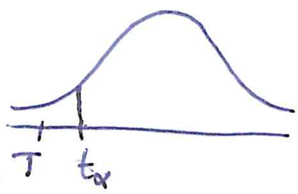
$= 0{,}515.$

$H_0: \mu_x - \mu_y = 0 \qquad H_1: \mu_x - \mu_y < 0$

$$T = \frac{\bar{x} - \bar{y}}{s_p \sqrt{\frac{1}{10} + \frac{1}{10}}} = \frac{42{,}14 - 43{,}23}{\sqrt{0{,}515\left(\frac{1}{10} + \frac{1}{10}\right)}} \approx -3{,}39.$$

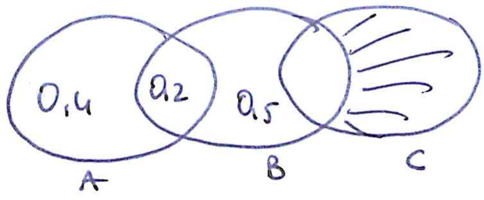
$\alpha = 0{,}05$, $df = 10+10-2 = 18$

$\Rightarrow t_\alpha = -1{,}734.$



$-3{,}39 < -1{,}734 \Rightarrow T \in$ Rejection region $\Rightarrow$ we reject the null hypothesis. Hence, we conclude that, on average, the new machine packs faster than the old machine.

3.



$P(A) = 0{,}4$, $P(B) = 0{,}5$

A and B are independent $\Rightarrow p(A \cap B) = p(A) \cdot p(B) = 0{,}4 \cdot 0{,}5 = 0{,}2$

A and C are disjoint $\Rightarrow p(A \cap C) = 0.$

Using the formula:

$$\underbrace{P(A \cup B \cup C)}^{=0{,}9} = P(A) + P(B) + P(C) - P(A \cap B) - \overbrace{P(A \cap C)}^{=0} - P(B \cap C) + \overbrace{p(A \cap B \cap C)}^{=0}$$

Since $A \cap C = \phi \Rightarrow A \cap B \cap C = \phi \Rightarrow p(A \cap B \cap C) = 0.$

$$\Rightarrow 0{,}9 = 0{,}4 + 0{,}5 + P(C) - 0{,}2 - p(B \cap C)$$

$$= 0{,}7 + P(C) - P(B \cap C)$$

$$\Rightarrow P(C) - P(B \cap C) = P(B^c \cap C) = 0{,}2.$$

This can also be done using Venn-diagram.

Or, by venndiagram $p(A \cup B \cup C) = p(A \cup B) + p(B^c \cap C)$

(since $A \cap C = \phi$)

$$\Rightarrow p(B^c \cap C) = p(A \cup B \cup C) - p(A \cup B)$$

$$= p(A \cup B \cup C) - (p(A) + p(B) - p(A \cap B))$$

$$= 0{,}9 - (0{,}4 + 0{,}5 - 0{,}2) = 0{,}2.$$

4. $m_X(t) = (1-t)^{-2}$

$m'_X(t) = -2(1-t)^{-3}(-1) = 2(1-t)^{-3}$

$\Rightarrow E[X] = m'_X(0) = 2.$ ①

$m''_X(t) = -6(1-t)^{-4}(-1) = 6(1-t)^{-4}$

$\Rightarrow m''_X(0) = 6$ ①

$\Rightarrow V[X] = m''_X(0) - (m'_X(0))^2 = 6 - 2^2 = 2.$

5. $f(x) = \frac{1}{39}x^2. \qquad 2 \le x \le 5$

(a) if $x < 2$, $F(x) = 0$.

if $2 \le x < 5$, $F(x) = \int_{-\infty}^{x} f(t)dt = \int_2^x \frac{1}{39} t^2 dt = \frac{1}{39}\frac{t^3}{3}\Big]_2^x = \frac{x^3}{117} - \frac{8}{117}.$

if $x \geq 5$, $F(x) = 1$

$$\Rightarrow F(x) = \begin{cases} 0 & \text{for } x < 2 \\ \frac{x^3}{117} - \frac{8}{117} & \text{for } 2 \le x < 5 \\ 1 & \text{for } x \geq 5 \end{cases}$$

b) (i) $P(X \le 1) = F(1) = 0$

(ii) $P(X \geq 4) = 1 - P(X < 4) = 1 - F(4) = 1 - \left(\frac{4^3}{117} - \frac{8}{117}\right)$

$= 1 - \frac{56}{117} = \frac{61}{117}.$

6. (a) $\hat{\mu}_{Y|x} = b_0 + b_1 x$

$$b_1 = \frac{n\sum x_i y_i - \sum x_i \sum y_i}{n\sum x_i^2 - (\sum x_i)^2} = \frac{12\cdot 6734 - 64\cdot 1041}{12\cdot 418 - 64^2} = 15{,}42$$

$$b_0 = \bar{y} - b_1\bar{x}$$

$$= \frac{\sum y_i}{12} - 15{,}42\frac{\sum x_i}{12} = \frac{1041}{12} - 15{,}42\left(\frac{64}{12}\right) = 4{,}51$$

$$\Rightarrow \hat{\mu}_{Y|x} = 4{,}51 + 15{,}42\,x.$$

(b) 95% CI on β_0:

$$b_0 \pm t_{0{,}975}\, s\,\frac{\sqrt{\sum x_i^2}}{\sqrt{n S_{xx}}}$$

$$S_{xx} = \left(n\sum x_i^2 - (\sum x_i)^2\right)/n = (12\cdot 418 - 64^2)/12$$
$$\doteq 76{,}67.$$

$$s^2 = \frac{SSE}{n-2} = \frac{S_{yy} - b_1 S_{xy}}{10}$$

$$S_{yy} = \left(n\sum y_i^2 - (\sum y_i)^2\right)/n = (12\cdot 108805 - 1041^2)/12$$
$$= 18498{,}25$$

$$S_{xy} = \left(n\sum x_i y_i - \sum x_i \sum y_i\right)/n = (12\cdot 6734 - 64\cdot 1041)/12$$
$$= 1182$$

$$\Rightarrow s^2 = \frac{18498{,}25 - 15{,}42\cdot 1182}{10} = 27{,}49.$$

$df = 10$. $t_{0{,}975} = 2{,}228$

$$\Rightarrow \text{C.I} = \left(4{,}51 \pm 2{,}228\cdot\sqrt{\frac{27{,}49\cdot 418}{12\cdot 76{,}67}}\right) = (-3{,}36, 12{,}4)$$

7 (a)

X	0	1	3	4
p_x	0,2	0,4	0,1	c

$\sum p_x = 1 \Rightarrow 0{,}2+0{,}4+0{,}1+c=1$

$\Rightarrow c = 1-0{,}2-0{,}4-0{,}1 = 0{,}3.$

(b) $P(XY>1)$

$XY>1$ if $Y=1$ and $X=3$ or $X=4$

$\Rightarrow P(XY>1) = P(X=3, Y=1) + P(X=4, Y=1)$

$= P(X=3)\,P(Y=1) + P(X=4)\,P(Y=1)$ since X and Y are independent

$= 0{,}1 \cdot 0{,}3 + 0{,}3 \cdot 0{,}3 = 0{,}03 + 0{,}09 = 0{,}12.$

8: $X \sim N(3, 16).$

Let $Y = 2X-5$ $\quad E[Y] = 2E[X]-5 = 6-5=1$

$V[Y] = 4V[X] = 4 \cdot 16 = 64.$

and Y is normally distributed

$\Rightarrow Y \sim N(1, 64).$

$P(2X-5>0) = P(Y>0) = P\Big(\underbrace{\frac{Y-1}{8}}_{Z} > \frac{-1}{8}\Big) = 1-P(Z \leq -\frac{1}{8})$

$= 1-P(Z \leq -0{,}125) \simeq 0{,}55.$

between 0,54776 and 0,55172.

9 (a) $e^{5(e^t-1)}$ (b) $0{,}235 = \frac{\binom{44}{4}\binom{56}{3}}{\binom{100}{7}}$ (c) 0,5

(d) 0,395 (e) Exponential (f) Geometric