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Tentamen

Datum: 191029 kl. 8.30–12.30
Telefonvakt: Felix Held
5325

MVE055/MSG810 Matematisk statistik och diskret matematik

Hjälpmedel: Chalmers godkända miniräknare och Betaboken.
Tentan rättas och bedöms anonymt. **Skriv tentamenskoden tydligt på placeringlista och samtliga inlämnade papper.** Fyll i omslaget ordentligt.
För godkänt på tentan krävs 12 poäng på tentamen.
För betyg VG för GU studenter krävs 22 poäng.
För betyg 4 resp. 5 för Chalmers studenter krävs dessutom 18 resp. 24 poäng.
Alla svar ska vara motiverade.
Uppgift 9 finns på separat blad på vilket lösning och svar kan skrivas. Detta blad inlämnas tillsammans med övriga lösningar.

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1. The experiment consists of rolling a fair die several times. Answer the following questions. (3p)

(a) Find the probability of getting number 5 for the first time at the sixth roll.
Geometric distribution with $p = \frac{1}{6}$. $P(X = 6) = (\frac{5}{6})^5 \frac{1}{6} = \frac{5^5}{6^6}$

(b) Find the probability of getting number 5 for the third time at the sixth roll.
Negative binomial with $r = 3$ and $p = \frac{1}{6}$. $P(X = 6) = \binom{5}{2}(\frac{1}{6})^3(\frac{5}{6})^3$.

(c) Find the probability of getting number 5 at least once if we roll the die exactly 10 times.
Binomial with $n = 10$ and $p = \frac{1}{6}$.
$P(X \geq 1) = 1 - P(X = 0) = 1 - (\frac{5}{6})^{10}$.

2. Let A and B be two events such that $P(A) = 0.2$, $P(B) = 0.3$ and $P(A|B) = 0.6$. (3p)

(a) Are A and B independent? Motivate your answer.
Since $P(A|B) \neq P(A)$ $(P(B) \neq 0)$, then A and B are not independent.

(b) Are A and B disjoint? Motivate your answer.
$P(A \cap B) = P(A|B)P(B) = 0.6(0.3) = 0.18 \neq 0$ then A and B are not disjoint.

(c) Compute $P(A \cup B)$.
$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.2 + 0.3 - 0.18 = 0.32$.

3. In 1980, out of 700 men aged between 20 and 34, 130 were found to be overweight. Whereas, in 1990, out of 750 men aged between 20 and 34, 160 were found to be overweight.

(a) Find an unbiased point estimate to p_1, the proportion of overweight men in 1980, and an unbiased point estimate to p_2, the proportion of overweight men in 1990, based on the samples given above. (1p)
An unbiased estimate for p_1 is $\hat{p}_1 = \frac{130}{700}$.

An unbiased estimate for p_2 is $\hat{p}_2 = \frac{160}{750}$.

(b) Find an unbiased point estimate for $p_1 - p_2$. (0.5p)
An unbiased estimate for $p_1 - p_2$ is $\hat{p}_1 - \hat{p}_2 = \frac{130}{700} - \frac{160}{750}$.

(c) Find a 95% confidence interval on the difference of proportions.
(Hint: The estimator for $p_1 - p_2$ is $\mathcal{N}(p_1 - p_2, \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2})$ where n_1 and n_2 are the corresponding sample sizes. A 95% C.I. is given by (1.5p)

$$\hat{p}_1 - \hat{p}_2 \pm z_{0.025}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

$z_{0.025} = 1.96$. The cofidence interval is given by

$$(-0.0687, 0.0135)$$

(d) Could you conclude based on the confidence interval computed in (c) that, for men 20-34 years old, a higher percentage were overweight in 1990 than 10 years earlier? Motivate your answer. (1p)
No since the interval contains 0.

4. The amount of fuel that a vessel has in its tank at the beginning of a voyage has a normal distribution with mean 1200 litres and standard deviation 80 litres. The amount which it will use on the voyage has a normal distribution with mean 800 litres and standard deviation 50 litres. Find the probability that it has less than 500 litres in the tank at the end of the voyage. (3p)
Let X be the amount of fuel at start and Y be the amount of fuel consumed. $X \sim \mathcal{N}(1200, 80^2)$ and $Y \sim \mathcal{N}(800, 50^2)$.
$X - Y$ is the amount of fuel at the end. $X - Y \sim \mathcal{N}(1200 - 800, 80^2 + 50^2)) = \mathcal{N}(400, 8900)$.
$P(X - Y < 500) = P(Z < \frac{500-400}{\sqrt{8900}}) = P(Z < 1.06) = 0.8554$.

5. Let X be a random variable with moment generating function

$$m_X(t) = (pe^t + q)^{10}$$

where p is constant between 0 and 1 and $q = 1 - p$.
Use $m_X(t)$ to find the expected value $E[X]$ and the variance $V[X]$ in terms of p. (3p)
$E[X] = m'_X(0)$.
$m'_X(t) = 10(pe^t + q)^9 pe^t$.
Then $E[X] = m'_X(0) = 10(pe^0 + q)^9 pe^0 = 10(p + q)p = 10p$ since $p + q = 1$.
$V[X] = E[X^2] = E[X]^2$.
$m''_X(t) = 10(9(pe^t + q)^9 pe^t)pe^t + 10(pe^t + q)^9 pe^t$.
Then $E[X^2] = m''_X(0) = 90(pe^0 + q)pe^0pe^0 + 10(pe^0 + q)^9 pe^0 = 90p^2 + 10p$.
Hence, $V[X] = 90p^2 + 10p - 100p^2 = 10p - 10p^2 = 10p(1 - p)$.

6. The following table consists of one student athlete's time (in minutes) to swim 2000 yards and the student's heart rate (beats per minute) after swimming on a random sample of 10 days:

Swim time	34.12	35.72	34.72	34.05	34.13	35.73	36.17	35.57	35.37	35.57
Heart rate	144	152	124	140	152	146	128	136	144	148

Denote by X the variable that gives the Swim time and Y the variable that gives the heart rate. The following information can be calculated using the above table:

$$\sum_{i=1}^{10} x_i = 351.15 \qquad \sum_{i=1}^{10} x_i^2 = 12\ 336.22 \qquad \sum_{i=1}^{10}(x_i - \overline{x})^2 = 5.5869$$

$$\sum_{i=1}^{10} y_i = 1\ 414 \qquad \sum_{i=1}^{10} y_i^2 = 200\ 756 \qquad \sum_{i=1}^{10}(y_i - \overline{y})^2 = 816.4$$

$$\sum_{i=1}^{10} x_i y_i = 49644.26 \qquad \sum_{i=1}^{10}(x_i - \overline{x})(y_i - \overline{y}) = -8.35$$

(a) Find the regression line $\hat{\mu}_{Y|X} = b_0 + b_1 x$ of the heart rate in terms of swim time. (2p)

$b_1 = \frac{S_{xy}}{S_{xx}} = \frac{-8.35}{5.5869} = -1.5$

$b_0 = \overline{y} - b_1\overline{x} = \frac{1414}{10} + 1.5(\frac{351.15}{10}) = 194.1$

Hence, the equation of the regression line is given by

$$\mu_{Y|X} = 194.1 - 1.5x$$

(b) Is there a significant linear relation between the heart rate in terms of swim time? Test $H_1 : \beta_1 \neq 0$ at the 0.1 level of significance. (3p)

$H_0 : \beta_1 = 0$ and $H_1 : \beta_1 \neq 0$. We have a two tailed test. $T = \frac{b_1}{s/\sqrt{S_{xx}}}$ follows a T distribution with degrees of freedom $n-2$.

$s^2 = \frac{SSE}{n-2} = \frac{S_{yy} - b_1 S_{xy}}{n-2} = 100.48$. Then $s = 10.0245$.

$t_{\alpha/2} = t_{0.05} = 1.86$.

$T = \frac{-1.5}{10.0245/\sqrt{5.5869}} = -0.3524$. Since $-1.86 < -0.3524 < 1.86$, then T is not in the rejection region and we cannot reject the null hypothesis. Hence, there is no significance relation between X and Y.

7. The purpose of this exercise is to find how many integer solutions the equation

$$y_1 + y_2 + y_3 = n \qquad (1)$$

has, where $y_1 \geq 2$ and $0 \leq y_2 \leq 5$, using generating functions.

(a) Prove that the generating function corresponding to the problem can be written as $f(x) = \frac{x^2(1-x^6)}{(1-x)^3}$. (2p)

$$\begin{aligned} f(x) &= (x^2 + x^3 + \cdots)(1 + x + x^2 + \cdots + x^5)(1 + x + x^2 + \cdots) \\ &= \frac{x^2}{1-x}\frac{1-x^6}{1-x}\frac{1}{1-x} \\ &= \frac{x^2(1-x^6)}{(1-x)^3} \end{aligned}$$

(b) How many integer solutions does equation (1) have for (2p)
(Hint: Use the common generating functions given on the next page).

i. n=3?
ii. n=15?

$f(x) = x^2(1-x^6)\sum_{n\geq 0}\binom{n+2}{2}x^n.$

i. The coefficient of x^3 is $\binom{3}{2} = 3$, then there are 3 integer solutions for $n = 3$, (namely 2+0+1=3, 2+1+0=3 and 3+0+0=3).
ii. The coefficient of x^{15} is $\binom{15}{2} - \binom{9}{2}$.

8. Let

$$f(x) = cx^2$$

for $0 \leq x \leq 6$.

(a) Find the value of c that makes $f(x)$ a density function for a continuous random variable X. (2p)
$f(x)$ is a density function if $f(x) \geq 0$ and $\int_0^6 f(x)dx = 1$.
$\int_0^6 f(x)dx = \int_0^6 cx^2 = c[\frac{x^3}{3}]_0^6 = 72c = 1$. Therefore $c = \frac{1}{72}$.

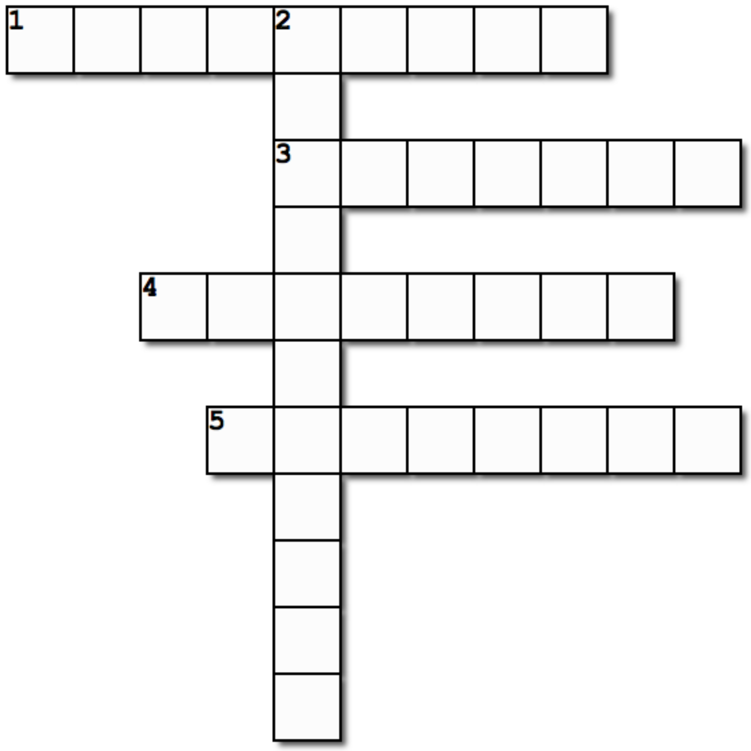
(b) Find the probability that $X \leq 5$ given that $4 \leq X \leq 7$. (2p)

$$\begin{aligned}
P(X \leq 5 | 4 \leq X \leq 7) &= \frac{P(\{X \leq 5\} \cap \{4 \leq X \leq 7\})}{P(4 \leq X \leq 7)} \\
&= \frac{P(4 \leq X \leq 5)}{P(4 \leq X \leq 6)} \\
&= \frac{\int_4^5 cx^2dx}{\int_4^6 cx^2dx} \\
&= \frac{5^3 - 4^3}{6^3 - 4^3} = 0.4013.
\end{aligned}$$

VÄND!

Anonym kod	MVE055/MSG810 191029	sid.nummer	Poäng

9. The following crossword consists of five words enumerated from 1 to 5 and whose clues are given below. Solve the crosswords or write your answers on the answer sheet. (1p)



1. Roll a fair die and denote by X the random variable that gives the number of rolls needed until we get 5 for the first time. Then X has a GEOMETRIC distribution.
2. Let X be the random variable that gives the time of the occurence of the first event in a Poisson process. Then X has a EXPONENTIAL distribution.
3. If X_1 and X_2 are random variables having Poisson distributions, then $X_1 + X_2$ has a POISSON distribution.
4. Roll a fair die 5 times and denote by X the number of times number 6 appears, then X has a BINOMIAL distribution.
5. Let $\hat{\theta}$ be a point estimator for a parameter θ. If $E[\hat{\theta}] = \theta$, then $\hat{\theta}$ is said to be UNBIASED.

Lycka till!

Common generating functions

(a) $\sum_{n=0}^{\infty} (cx)^n = \frac{1}{1-cx}$, for $|x| < 1$.

(b) $\sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$.

(c) $\sum_{n=0}^{\infty} \binom{k+n}{k} x^n = \frac{1}{(1-x)^{k+1}}$.

T-distribution table

df	0.60	0.75	0.90	0.95	0.975	0.99	0.995	0.999
6	0.265	0.718	1.440	1.943	2.447	3.143	3.707	5.208
7	0.263	0.711	1.415	1.895	2.365	2.998	3.499	4.785
8	0.262	0.706	1.397	1.860	2.306	2.896	3.355	4.501
9	0.261	0.703	1.383	1.833	2.262	2.821	3.250	4.297
10	0.260	0.700	1.372	1.812	2.228	2.764	3.169	4.144

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997