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Applied Mathematics and Statistics
Chalmers and GU

MVE055 / MSG810 Matematisk statistik och diskret matematik

Exam 7 January 2019, 14:00 - 18:00

Allowed aids: Chalmers-approved calculator
and one (two-sided) A4 sheet of paper with your own notes.
Total number of points: 30. To pass, at least 12 points are needed.
Note: All answers should be motivated.

1 Solutions

1. We know $\bar{X} = 94.32$, $s^2 = 1.5$ and $n = 16$. We want to test H_0: the average lenght is 95 cm, against H_1: the average length is not equal to 95 cm (twosided test) at level $\alpha = 0.01$.

   (a) Consider the test statistic $T = \frac{\bar{X}-\mu}{\frac{s}{\sqrt{n}}}$. Under the null hypothesis, T follows a T distribution with $n - 1 = 15$ degrees of freedom. We consider a level $\alpha = 0.01$, hence we need to find $t_{0.005}$ as defined by the following equation

   $$P(T \leq -t_{0.005}) = 0.005.$$

   Looking at the statistical tables, we find $t_{0.005} = 2.947$. The observed value of the test statistic is $T = \frac{94.32-95}{\frac{\sqrt{1.5}}{\sqrt{(16)}}} = -2.22$. Since the value -2.22 is not outside the interval $[-2.947, 2.947]$ we fail to reject the null hypothesis.

   (b) Consider the test statistic $Z = \frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}$. Under the null hypothesis, Z follows a standard normal distribution. We consider a level $\alpha = 0.01$, hence we need to find $t_{0.005}$ as defined by the following equation

   $$P(Z \leq -z_{0.005}) = 0.005.$$

   Looking at the statistical tables, we find $z_{0.005} = 2.575$. The observed value of the test statistic is $Z = \frac{94.32-95}{\frac{\sqrt{1.2}}{\sqrt{(16)}}} = -2.483$. Since the value -2.483 is not outside the interval $[-2.575, 2.575]$ we fail to reject the null hypothesis.

2. We know $P(A \cup B) = 0.626, P(A \cap B) = 0.144$. We have the following system of equation for $P(A)$ and $P(B)$ using the fact that A and B are independent

   $$\begin{cases} P(A \cup B) = P(A) + P(B) - P(A \cap B) \\ P(A)P(B) = P(A \cap B) \end{cases}$$

We can find $P(B)$ for one of the two equations and substitute its value in the other to find a second order equation for $P(B)$. The equation has two solutions $P(B) = 0.45$ or $P(B) = 0.32$, which lead to $P(A) = 0.32$ and $P(A) = 0.45$, respectively. Since $P(A) > P(B)$ by assumption, the only possible solution is given by $P(A) = 0.45$, $P(B) = 0.32$.

3. (a) The scatterplot shows a strong linear relationship which supports the use of the linear regression model. We expect the correlation coefficient to be positive, as the extension produced increases as the force applied increases.

   (b) Let us define the following quantities

$$\bar{x} = \frac{1}{15}\sum_{i=1}^{15} x_i = 53.2$$

$$\bar{y} = \frac{1}{15}\sum_{i=1}^{15} y_i = 42.867$$

$$S_{xx} = \sum_{i=1}^{15} x_i^2 - \frac{(\sum_{i=1}^{15} x_i)^2}{15} = 20586.4$$

$$S_{yy} = \sum_{i=1}^{15} y_i^2 - \frac{(\sum_{i=1}^{15} y_i)^2}{15} = 14435.7$$

$$S_{xy} = \sum_{i=1}^{15} x_i y_i - \frac{(\sum_{i=1}^{15} x_i)(\sum_{i=1}^{15} y_i)}{15} = 17024.4$$

   The point estimates for the slope β_1 and the intercept β_0 of the regression line are

$$\hat{\beta}_1 = \frac{S_{xy}}{S_{xx}} = 0.82697$$

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x} = -1.1278.$$

   (c) We have

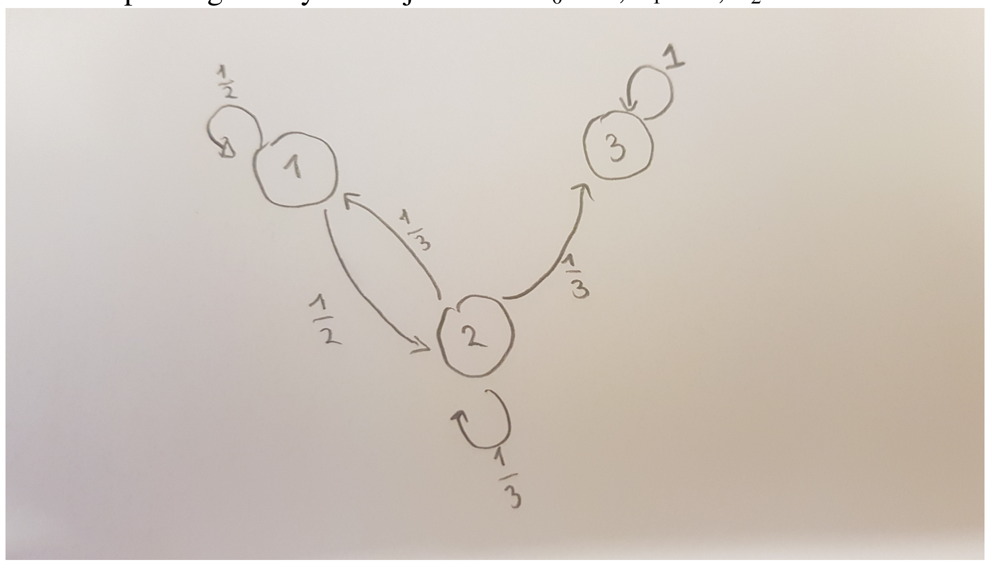
$$SSE = S_{yy} - \hat{\beta}_1 S_{xy} = 357.07$$

   The coefficient of determination is given by

$$r^2 = 1 - \frac{SSE}{S_{yy}} = 0.9753.$$

   We note that 97.53% of the observed variation in the extension can be attributed to the linear relationship between force applied and extension of the spring.

4. Without loss of generality, let $X_n, n = 0, 1, 2, ...$ be the Markov chain and let $1, 2, 3$ be the possible states. We want to find $P(X_2 = 1|X_0 = 1)$. The following picture shows that the only ways to be at state 1 in two steps are given by the trajectories: $X_0 = 1, X_1 = 1, X_2 = 1$



or $X_0 = 1, X_1 = 2, X_2 = 1$.

$$P(X_2 = 1|X_0 = 1) = P(X_2 = 1|X_1 = 1)P(X_1 = 1|X_0 = 1) + P(X_2 = 1|X_1 = 2)P(X_1 = 2|X_0 = 1)$$
$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{12} = 0.4166.$$

5. (a) For a random variable X, the moment generating function $m(t)$ is defined as $m(t) = \mathbb{E}[e^{tX}]$ (provided that such expectation is finite for all t in some open interval).

   (b) If X follows a Poisson distribution with parameter λ, we have

$$m(t) = \mathbb{E}[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} =$$
$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} e^{-e^t\lambda} e^{e^t\lambda} = e^{-\lambda} e^{e^t\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!} e^{-e^t\lambda} =$$
$$= e^{-\lambda} e^{e^t\lambda} = e^{\lambda(e^t-1)}.$$

   (c) We have

$$\mathbb{E}[X] = \frac{dm(t)}{dt}|_{t=0} = e^{\lambda(e^t-1)}\lambda e^t|_{t=0} = \lambda.$$

6. Define the events: M=mathematics student, P=physics student, E= engineering student, Pass = student pass the exam. Using Bayes' theorem we have

$$P(M|Pass) = \frac{P(Pass|M)P(M)}{P(Pass)} = \frac{P(Pass|M)P(M)}{P(Pass|M)P(M) + P(Pass|P)P(P) + P(Pass|E)P(E)} =$$
$$= \frac{0.99 \cdot 0.15}{0.99 \cdot 0.15 + 0.75 \cdot 0.25 + 0.6 \cdot 0.6} = \frac{0.1485}{0.696} = 0.2134.$$