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Applied Mathematics and Statistics
Chalmers and GU

MVE055 / MVE051 / MSG810 Matematisk statistik och diskret matematik

Exam 29 August 2018, 14:00 - 18:00

Allowed aids: Chalmers-approved calculator
and one (two-sided) A4 sheet of paper with your own notes.
Total number of points: 30. To pass, at least 12 points are needed.
Note: All answers should be motivated.

Solutions

1. (a) Since $P(X = k) = \frac{1}{4}, k \in \{-3, -1, 1, 3\}$ then $P(Y = r) = \frac{1}{2}, r \in \{1, 9\}$. We can compute $\mathbb{E}[X] = 0$ and similarly $\mathbb{E}[X^3] = 0$ since X is symmetric around 0. We observe now that, since $Y = X^2$, $XY = XX^2 = X^3$, implying that $\mathbb{E}[XY] = \mathbb{E}[X^3] = 0$. Thus,

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X^3] - 0 = 0.$$

Hence, X and Y are uncorrelated.

(b) X, Y are not independent since, for example,

$$P(X = -1, Y = 9) = 0 \neq P(X = -1)P(Y = 9).$$

2. Chebychev's inequality: Let X be a random variable such that $\mathbb{E} = \mu$, $Var(X) = \sigma^2$. If $0 < \sigma^2 < \infty$, then for any $a > 0$ it holds

$$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}.$$

Proof: Define $Y = |X - \mu|$. For any $a > 0$ we define

$$Z = \begin{cases} a^2 & Y \geq a \\ 0 & \text{otherwise} \end{cases}$$

By definition $Z \leq Y^2$, which implies

$$\mathbb{E}[Z] \leq \mathbb{E}[Y^2]. \tag{1}$$

Moreover, we observe

$$\mathbb{E}[Z] = a^2 P(Y \geq a) = a^2 P(|X - \mu| \geq a)$$

and

$$\mathbb{E}[Y^2] = \mathbb{E}[|X-\mu|^2] = Var(X) = \sigma^2.$$

Replacing the above equations into Equation 1 we obtain

$$a^2 P(|X-\mu| \geq a) \leq Var(X)$$

which can be rewritten as

$$P(|X-\mu| \geq a) \leq \frac{Var(X)}{a^2} = \frac{\sigma^2}{a^2}.$$

3. (a) Given that $P(X \geq x+1|X \geq x) = 1-p$, for all $x = 1, 2, 3, ...$ we have by definition of conditional probability, for all $x = 1, 2, 3, ...,$

$$1-p = P(X \geq x+1|X \geq x) = \frac{P(X \geq x+1, X \geq x)}{P(X \geq x)} = \frac{P(X \geq x+1)}{P(X \geq x)}.$$

(b) Using the previous point, for all $x = 1, 2, 3, ...,$

$$P(X \geq x+1) = (1-p)P(X \geq x) = (1-p)^2 P(X \geq x-1) = ... = (1-p)^x P(X \geq 1) = (1-p)^x,$$

where $P(X \geq 1) = 1$ since X is always greater or equal to 1.

(c)

$$P(X = x) = P(X \geq x) - P(X \geq x+1) = (1-p)^x - (1-p)^{x+1} = (1-p)^x(1-(1-p)) = p(1-p)^x.$$

X has a Geometric distribution with parameter p.

4. (a) TRUE: the minimum value U can assume is a. Thus the minimum value V can assume is $s + at$. Similarly, the maximum value of V is $s + bt$. For any value $v \in [s + at, s + bt]$ the density of V is

$$f_V(v) = f_U(\frac{v-s}{t})|\frac{1}{t}| = \frac{1}{t(b-a)}$$

which is the density of a uniform random variable on the interval $[s + at, s + bt]$.

(b) FALSE: see previous motivation.

(c) FALSE: The expected value of U is $\mathbb{E}[U] = \frac{b+a}{2}$. Using the linearity of expectation, we obtain

$$\mathbb{E}[V] = \mathbb{E}[s + tU] = s + t + \mathbb{E}[U] = s + t\frac{b+a}{2}$$

5. The $100(1-\alpha)\%$ standard confidence interval for the mean μ is given by

$$I_\alpha = \left[\bar{X} - z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} + z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\right],$$

where $\bar{X}$ denotes the sample mean and $z_{\frac{\alpha}{2}}$ the point such that $P(N(0,1) \geq z_{\frac{\alpha}{2}}) = \frac{\alpha}{2}$. The interval I_α has width

$$\text{width of } I_\alpha = 2z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}. \quad (2)$$

(a) If α increases, the width of I_α decreases. In fact, when α increases, we are requiring a lower probability that the confidence interval will contain the mean μ. Thus, the width of I_α will be smaller.

(b) If σ^2 decreases then the width decreases. In particular, when the variance decreases by a factor 4, the width of the confidence interval considered will halve.

(c) If n doubles, the width of the confidence interval considered here will decrease by a factor $\sqrt{2}$.

6. (a) The transition matrix P in canonical form is (rows/columns refers in order to the states $1, 2, 0, 3$ in this order)

$$P = \begin{bmatrix} 0 & 0.25 & 0.75 & 0 \\ 0.75 & 0 & 0 & 0.25 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Q & R \\ 0 & I \end{bmatrix}$$

(b) The fundamental matrix is $N = (I - Q)*^{-1} = \begin{bmatrix} \frac{16}{13} & \frac{4}{13} \\ \frac{12}{13} & \frac{16}{13} \end{bmatrix}$. To obtain the probability of bankrupt, i.e. the probability of being absorbed in the state 0, we need to compute

$$B = NR = \begin{bmatrix} \frac{16}{13} & \frac{4}{13} \\ \frac{12}{13} & \frac{16}{13} \end{bmatrix} \begin{bmatrix} \frac{3}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{12}{13} & \frac{1}{13} \\ \frac{9}{13} & \frac{4}{13} \end{bmatrix}.$$

Since the chain starts in the state 2, the probability of being absorbed in 0 is $\frac{9}{13} = 0.692$.