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Mathematical Statistics
Chalmers and GU

MVE055/MSG810 Mathematical statistics and discrete mathematics
MVE055/MSG810 Matematisk statistik och diskret matematik

Re-exam: 4 January 2016, 14:00 - 18:00, M building
Examiner and jour: Petter Mostad, telphone 0707163235, will be available for questions 15:00 and 17:00
Allowed aids: A Chalmers-approved calculator and at most one (double-sided) A4 page of personal notes. Some tables of statistical distributions are provided.
Grading: This exam can give a maximum of 30 points. For full points, each answer must be motivated. Points from VLE tests autumn 2015 will be added. To pass, a minimum of 12 points is needed.
Language: You may write your answers in either English or Swedish.

1. (6 points) Let X be a random variable that takes on the four values 0, 1, 2, and 3 with probabilities 0.4, 0.3, 0.2, and 0.1, respectively. Let Y be an independent random variable that takes on the three values 0, 1, and -1 with probabilities 0.4, 0.3, and 0.3, respectively. Calculate:

   (a) $\mathrm{P}(X > 1)$

   (b) $\mathrm{P}(XY > 0)$

   (c) $\mathrm{P}(X + Y = 1)$

   (d) $\mathrm{E}(X)$

   (e) $\mathrm{E}(XY)$

   (f) $\mathrm{Var}(Y)$

2. (4 points) Explain in your own words the Method of Moments: Its purpose and its idea. Then, consider the Negative Binomial distribution with parameters r and p, where r is a positive integer and $p \in (0, 1)$. A random variable with this distribution has expectation r/p and variance $r(1 - p)/p^2$. Assume $X_1, X_2, \dots, X_n$ is a random sample from a Negative Binomial distribution where both r and p are unknown parameters. Use the Method of Moments to obtain estimators for r and p.

3. (2 points) State the Central Limit Theorem as precisely as you can, and mention one example showing the practical usefulness of this theorem.

4. (3 points) Let X be a random variable with a standard normal distribution.

   (a) Write down its density function.

(b) Compute the moment generating function of X (i.e., don't just write it down).

(c) Use the moment generating function to compute the first, second, and third moments of X.

5. (4 points) Assume the observations 12.4, 18.5, 14.6, 16.4, 12.9, 11.9, 16.9, 19.5, and 22.6 are a random sample from a normal distribution with known standard deviation $\sigma = 3.7$.

   (a) Compute a confidence interval with confidence level 95% for the expectation μ of the normal distribution.

   (b) Assume we continue to sample 41 additional values, and then compute a new 95% confidence interval. What will be the length of this interval?

   (c) Assume we would like to continue to sample until we have a 99% confidence interval of length 1.1. How many values do we need to sample in total?

   (d) We now drop the assumption that we know the standard deviation of the distribution, but keep assuming the data is a random sample from some normal distribution. Compute a confidence interval for μ with confidence level 95% under these relaxed assumptions.

6. (4 points) Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables such that $P(X_i = 1) = p$ and $P(X_i = 0) = 1 - p$ for some p with $0 < p < 1$.

   (a) Compute the expectation and variance of X_i.

   (b) Compute the expectation and variance of

   $$\overline{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

   (c) Find an approximate distribution of $\overline{X}$ when n is large.

   (d) Use the above to derive a formula for an approximate confidence interval with a 95% confidence level for the parameter p.

7. (4 points) Assume you are given bivariate data $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, i.e., the coordinates of n points in the plane.

   (a) Explain what the simple linear regression model is and what the least squares solution is (you do not need to provide formulas): Explain the purpose and the idea.

   (b) Define

   $$S = \sum_{i=1}^{n} y_i - \widehat{y_i}$$

   where $\widehat{y_i}$ is the value such that the point $(x_i, \widehat{y_i})$ lies on the regression line. What is the value of S?

8. (3 points) Consider the following 6 matrices:

$$P_1 = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}, \quad P_2 = \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}, \quad P_6 = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{pmatrix}.$$

For each, determine whether the matrix is a transition matrix for a Markov Chain, and if it is, whether this chain is absorbing, whether it is ergodic (i.e., irreducible), and whether it is regular. Motivate each answer.