Lösningar till tentamen i Matematisk statistik och diskret matematik D2 (MVE055/MSG810).
Den 28 oktober 2014. These are sketches of the solutions.

1. *Proof of* $\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(X^2)}{a^2}$. For $a > 0$, let

$$I = \begin{cases} 1 & \text{if } |X| \geq a \\ 0 & \text{oterwise} \end{cases}$$

and note that,

$$I \leq \frac{X^2}{a^2}$$

Taking expectations of the preceding inequality yields

$$\mathbb{E}(I) \leq \frac{\mathbb{E}(X^2)}{a^2}$$

which, because $\mathbb{E}(I) = \mathbb{P}(|X| \geq a)$, proves the result. □

2. a) The moment generating function of a random variable X is defined as

$$M_X(t) = \mathbb{E}(e^{tX}) = \begin{cases} \sum_x e^{tx}\mathbb{P}(X = x), \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)dx. \end{cases}$$

b) $M_X(t) = e^{\lambda(e^t - 1)}$.

c) $\gamma_1 = \frac{1}{\lambda^{\frac{1}{2}}}$.

3. A confidence interval for μ is based on the fact that

$$\frac{\sqrt{n}(\bar{X} - \mu)}{S} \sim t_{n-1}$$

where t_{n-1} denotes the t distribution with $n-1$ degrees of freedom and

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

Let $t_{n-1,\alpha/2}$ denote that point beyond which the t distribution with $n-1$ degrees of freedom has probability $\alpha/2$. Since the t distribution is symmetric about 0, the probability to the left of $-t_{n-1,\alpha/2}$ is also $\alpha/2$. Then, by definition,

$$\mathbb{P}\Big(-t_{n-1,\alpha/2} \leq \frac{\sqrt{n}(\bar{X} - \mu)}{S} \leq t_{n-1,\alpha/2}\Big) = 1 - \alpha.$$

The inequality can be manipulated to yield

$$\mathbb{P}\Big(\bar{X} - \frac{S}{\sqrt{n}}t_{n-1,\alpha/2} \leq \mu \leq \bar{X} + \frac{S}{\sqrt{n}}t_{n-1,\alpha/2}\Big) = 1 - \alpha.$$

According to this equation, the probability that μ lies in the interval $\bar{X} \pm {}^{St_{n-1,\alpha/2}}/_{\sqrt{n}}$ is $1 - \alpha$. Note that this is random: The center is at the random point $\bar{X}$ and the width is proportional to S, which is also random. A second method of constructing confidence intervals is based on the large sample theory. According to this method

$$\bar{X} \pm {}^{Sz_{\alpha/2}}/_{\sqrt{n}}$$

is an approximate $100(1-\alpha)\%$ confidence interval.

4. Process $(Y_n)_{n\geq 0}$ is a Markov chain with state space $S = \{1, 2\}$, initial state 1 and matrix of transition probabilities

$$\begin{pmatrix} 3/7 & 4/7 \\ 1/11 & 10/11 \end{pmatrix}.$$