Lösningar till tentamen i Matematisk statistik och diskret matematik D2 (MVE055/MSG810).
Den 20 oktober 2012. These are sketches of the solutions.

1. Lösning:

a)

Proposition 1. *If X is a random variable that takes only nonnegative values, then, for any value $a > 0$,*

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

Proof of the Markov's inequality. For $a > 0$, let

$$I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{oterwise} \end{cases}$$

and note that, since $X \geq 0$,

$$I \leq \frac{X}{a}$$

Taking expectations of the preceding inequality yields

$$\mathbb{E}(I) \leq \frac{\mathbb{E}(X)}{a}$$

which, because $\mathbb{E}(I) = \mathbb{P}(X \geq a)$, proves the result. □

b)

Proposition 2. *If X is a random variable with finite mean μ and variance σ^2, then, for any value $k > 0$,*

$$\mathbb{P}\big(|X - \mu| \geq k\big) \leq \frac{\sigma^2}{k^2}$$

Proof of the Chebyshev's inequality. Since $(X-\mu)^2$ is a nonnegative random variable, we can apply Markov's inequality (with $a = k^2$) to obtain

$$\mathbb{P}\big((X-\mu)^2 \geq k^2\big) \leq \frac{\mathbb{E}\big((X-\mu)^2\big)}{k^2} \tag{1}$$

But since $(X-\mu)^2 \geq k^2$ if and only if $|X - \mu| \geq k$, Equation (1) is equivalent to

$$\mathbb{P}\big(|X - \mu| \geq k\big) \leq \frac{\mathbb{E}\big((X-\mu)^2\big)}{k^2} = \frac{\sigma^2}{k^2}$$

and the proof is complete. □

c)

$$\begin{aligned}\mathbb{P}(0 < X < 40) &= \mathbb{P}(-20 < X - 20 < 20) = \mathbb{P}\big(|X - 20| < 20\big) \\ &= 1 - \mathbb{P}\big(|X-20| \geq 20\big) \geq 1 - \frac{20}{20^2} \quad \text{by Chebyshev's inequality} \\ &= 1 - \frac{1}{20} = \frac{19}{20} = 0.95\end{aligned}$$

d) Chebyshev's inequality must be regarded as a theoretical tool rather than a practical method of estimation. Its importance is due to its universality, but no statements of great generality can be expected to yield sharp results in individual cases.

Lösning:

2. a) The generating function for the infinite series $\langle g_0, g_1, g_2, g_3, \ldots \rangle$ is the power series:

$$G(x) = g_0 + g_1 x + g_2 x^2 + g_3 x^3 + \ldots$$

Recall that the sum of an infinite geometric series is:

$$1 + z + z^2 + z^3 + \ldots = \frac{1}{1-z}$$

This equation does not hold when $|z| \geq 1$, but we don't worry about convergence issues. This formula gives closed-form generating function for the sequence $\langle 1, 1, 1, 1, \ldots \rangle$.

$$\langle 1, 1, 1, 1, \ldots \rangle \longleftrightarrow 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

Differentiate the generating function for an infinite sequence of 1's.

$$\frac{d}{dx}(1 + x + x^2 + x^3 + x^4 + \ldots) = \frac{d}{dx}\left(\frac{1}{1-x}\right)$$
$$1 + 2x + 3x^2 + 4x^2 + \ldots = \frac{1}{(1-x)^2}$$
$$\langle 1, 2, 3, 4, \ldots \rangle \longleftrightarrow \frac{1}{(1-x)^2}$$

The generating function for the sequence $\langle 1, 2, 3, 4, \ldots \rangle$ of positive integers is $\frac{1}{(1-x)^2}$.

b) Generating function are particulary useful for solving counting problems. Problems involving choosing items from a set often lead to nice generating functions by letting the coefficient of x^n be the number of ways to choose n items. Often we can translate the description of a counting problem directly into a generating funciton for the solution. For example, the generating function of binomial coefficients and the generating function for selecting items from a k-element set with repretition.

c) First construct a generating function for selecting egg bagels. We can select a set of 0 or 1 bagel(s) in 0 way (because at least two bagels of each kind are chosen), 2 bagels in one way, a set of 3 bagels in one way, and so forth. So we have:

$$E(x) = x^2 + x^3 + x^4 + \ldots = \frac{x^2}{1-x}$$

Similarly, the generating function for selecting plain bagels is:

$$P(x) = x^2 + x^3 + x^4 + \ldots = \frac{x^2}{1-x}$$

Now, we can select a set of 0 or 1 egg bagel(s) in 0 way (because at least two bagels of each kind are chosen), 2 salty bagels in one way, a set of 3 salty bagels in one way. However, we can not select more than three salty bagels, so we have the generating function:

$$S(x) = x^2 + x^3$$

The Convolution Rule says that the generating function for selecting from among all three kinds of bagels is:

$$E(x)P(x)S(x) = \frac{x^6 + x^7}{(1-x)^2}$$

which is the generating function for sequence $\langle \overbrace{0, 0, \ldots, 0}^{\text{6 zeroes}}, 1, 3, 5, 7, \ldots \rangle$. Thus we can select a dozen of bagels in 13 different ways.

3. Lösning:
   Since the sample size $n = 5$ is small and variance of an underlying normal distribution is unknown the Student's t-CI on mean should be used:

$$\bar{X} \pm t_{n-1,\frac{\alpha}{2}} \frac{S}{\sqrt{n}}$$

   where

$$S = \sqrt{\frac{1}{n-1}(X_i - \bar{X})^2} \quad \text{is a sample standard deviation.}$$

   Since $n = 5$, d.f.$=4$; therefore, from Table of the t-distribution, $t_{4,0.025} = 2.776$; $\bar{x} = 13.38$, $s^2 = 0.297$. We obtain a 95% confidence interval $(12.703, 14.057)$.