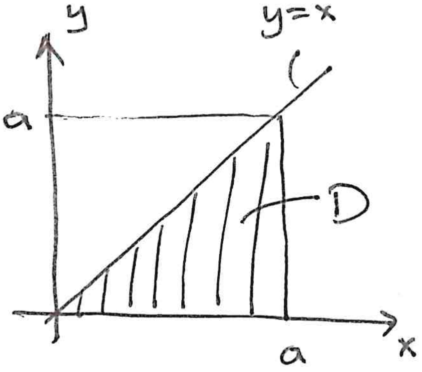
MVE041, Flervariabelsanalys Z

Lösningar tenta 1/6 – 20



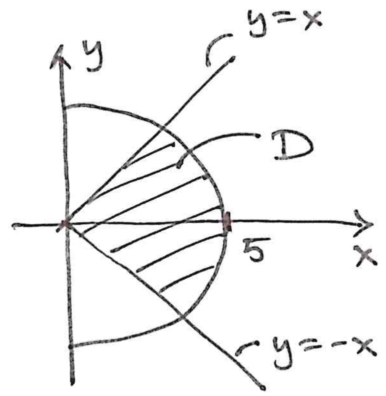
1(a) $\displaystyle\int_0^a\left(\int_y^a \cos(x^2)\,dx\right)dy = \iint_D \cos(x^2)\,dA =$

$\displaystyle = \int_0^a\left(\int_0^x \cos(x^2)\,dy\right)dx = \int_0^a x\cos(x^2)\,dx =$

$\displaystyle = \frac{1}{2}\left[\sin(x^2)\right]_0^a = \frac{1}{2}\sin(a^2) \overset{\text{vill}}{=} \frac{1}{2}$

$\Leftrightarrow \sin(a^2) = 1 \Rightarrow a^2 = \frac{\pi}{2} + 2\pi\cdot n,\ n = 0, 1, 2, \ldots$

$\therefore\ a = \sqrt{\frac{\pi}{2} + 2\pi\cdot n}\ ,\ n = 0, 1, 2, \ldots$



(b) $\displaystyle\iint_D y^2 e^{x^2+y^2}\,dA = \left\{\begin{matrix}\text{Polära}\\ \text{koord.}\end{matrix}\right\} =$

$\displaystyle = \int_{-\pi/4}^{\pi/4}\left(\int_0^5 r^2\sin^2\theta\cdot e^{r^2} r\,dr\right)d\theta =$

$\displaystyle = \int_{-\pi/4}^{\pi/4} \sin^2\theta\,d\theta \cdot \int_0^5 r^3 e^{r^2}\,dr = I_1\cdot I_2$

$\displaystyle I_1 = \int_{-\pi/4}^{\pi/4}\sin^2\theta\,d\theta = \int_{-\pi/4}^{\pi/4}\frac{1-\cos(2\theta)}{2}\,d\theta = \frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} =$

$\displaystyle = \frac{1}{2}\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right) - \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin\left(-\frac{\pi}{2}\right)\right) = \frac{\pi}{4} - \frac{1}{2}$

$\displaystyle I_2 = \int_0^5 r^2 e^{r^2}\cdot r\,dr = \left\{\begin{matrix} t = r^2, & r=0 \Leftrightarrow t=0 \\ dt = 2r\,dr, & r=5 \Rightarrow t=25\end{matrix}\right\} = \int_0^{25} t e^t \frac{dt}{2} =$

$\displaystyle = \left\{\begin{matrix}\text{part.}\\ \text{integr.}\end{matrix}\right\} = \frac{1}{2}\left(\left[te^t\right]_0^{25} - \int_0^{25} e^t\,dt\right) = \frac{1}{2}\left(25e^{25} - e^{25} + 1\right) =$

$$= 12e^{25} + \frac{1}{2}$$

$$\therefore \iint_D y^2 e^{x^2+y^2}\, dA = \left(\frac{\pi}{4} - \frac{1}{2}\right)\left(12e^{25} + \frac{1}{2}\right)$$

2. Låt $\mathbb{F}(x,y) = (x^3 + axy^2, y^3 + ax^2y)$

Vill beräkna: $\int_\gamma \mathbb{F} \cdot d\mathbb{r}$

$\frac{\partial F_2}{\partial x} = 2axy = \frac{\partial F_1}{\partial y}$ och $D_{\mathbb{F}} = \mathbb{R}^2$ enkelt sammanhängande

$\Rightarrow \mathbb{F}$ konservativ $\Rightarrow \exists \phi : \mathbb{R}^2 \to \mathbb{R}$ s.a. $\mathbb{F} = \nabla \phi$

$\frac{\partial \phi}{\partial x} = F_1 = x^3 + axy^2 \Rightarrow \phi(x,y) = \frac{x^4}{4} + \frac{1}{2}ax^2y^2 + g(y)$

$\Rightarrow y^3 + ax^2y = F_2 = \frac{\partial \phi}{\partial y} = ax^2y + g'(y) \Leftrightarrow g'(y) = y^3$

$\Rightarrow g(y) = \frac{1}{4}y^4 + C$

$\therefore \phi(x,y) = \frac{x^4}{4} + \frac{y^4}{4} + \frac{1}{2}ax^2y^2 + C = \frac{x^4 + 2ax^2y^2 + y^4}{4} + C$

$\int_\gamma \mathbb{F} \cdot d\mathbb{r} = \int_\gamma \nabla\phi \cdot d\mathbb{r} = \phi(\gamma_{slut}) - \phi(\gamma_{start}) =$

$= \phi(2,3) - \phi(0,3) = \frac{2^4 + 2a \cdot 2^2 \cdot 3^2 + 3^4}{4} + C - \left(\frac{3^4}{4} + C\right) =$

$= \frac{4 \cdot 4 + 2 \cdot 4 \cdot 9a}{4} + \frac{3^4}{4} - \frac{3^4}{4} = \underline{\underline{4 + 18a}}$

$\int_\gamma \mathbb{F} \cdot d\mathbb{r} = -50 \Leftrightarrow 4 + 18a = -50 \Leftrightarrow 18a = -54$

$\therefore a = -\frac{54}{18} = \underline{\underline{-3}}$

3. (a) Låt $f(x,y,z) = x^2 - y^2 - z^2$ och $g(x,y,z) = 3x^2 + 2y^2 + z^2$

Då är de givna ytorna nivåytor till f och g.

$\nabla f(x,y,z) = (2x, -2y, -2z)$, $\nabla g(x,y,z) = (6x, 4y, 2z)$

$\nabla f(1,1,2) = (2,-2,-4) \leftarrow$ normal till nivåytan $f = -4$

$\nabla g(1,1,2) = (6,4,4) \leftarrow$ normal till nivåytan $g = 9$

$\Rightarrow$ riktningsvektorn för tangentlinjen är parallell med:

$$\nabla f(1,1,2) \times \nabla g(1,1,2) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -2 & -4 \\ 6 & 4 & 4 \end{vmatrix} = (8, -32, 20) = 4(2,-8,5)$$

$$\therefore \ (x,y,z) = (1,1,2) + t(2,-8,5), \ t \in \mathbb{R}$$

(b) Låt $z = f(x,y) = \frac{1}{100}(50 - x^2 - y^2)$

Linjestycket från $(7,-1)$ till $(-1,3)$ har riktningsvektorn

$$u = (-1,3) - (7,-1) = (-8,4) = 4(-2,1)$$

$$\Rightarrow \hat{u} = \frac{u}{|u|} = \frac{1}{4\sqrt{(-2)^2 + 1^2}} 4(-2,1) = \frac{1}{\sqrt{5}}(-2,1)$$

Förändringen av höjden $z = f(x,y)$ längs riktningen $\hat{u}$ ges av:

$$D_{\hat{u}} f(x,y) = \hat{u} \cdot \nabla f(x,y) = \frac{1}{\sqrt{5}}(-2,1) \cdot \frac{1}{50}(-x,-y) =$$

$$= \frac{1}{50\sqrt{5}}(2x - y)$$

Räta linjen mellan $(7,-1)$ och $(-1,3)$:

$$k = \frac{3-(-1)}{-1-7} = -\frac{4}{8} = -\frac{1}{2} \Rightarrow$$

$\Rightarrow \ y = -\frac{1}{2}(x-(-1))+3 = -\frac{1}{2}x + \frac{5}{2} \Rightarrow$

$\Rightarrow D_{\hat{u}_1} f(x, -\frac{1}{2}x+\frac{5}{2}) = \frac{1}{50\sqrt{5}}\left(2x-(-\frac{1}{2}x+\frac{5}{2})\right) =$

$= \frac{1}{50\sqrt{5}}\left(\frac{5}{2}x - \frac{5}{2}\right) = \frac{1}{20\sqrt{5}}(x-1)$

Då $7 \geq x \geq -1$ så $D_{\hat{u}_1} f$ som störst då $x = 7$ och som minst då $x = -1$. $D_{\hat{u}_1} f = 0$ när det är plant, vilket sker då $x = 1 \Rightarrow$

$\Rightarrow y = -\frac{1}{2} + \frac{5}{2} = 2 \Rightarrow z = f(1,2) = \frac{1}{100}(50-1-4) = 0.45$

$\therefore$ Max lutning $\frac{3}{10\sqrt{5}}$ i by nr. 1

Min lutning $-\frac{1}{10\sqrt{5}}$ i by nr. 2

Fixapaus i $(1, 2, 0.45)$

4. Låt $g: \mathbb{R}^2 \to \mathbb{R}$ s.a. $f(x,y) = g(u,v) = g(y^2+x^2, y^2-x^2)$.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} g(u,v) = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} = 2x \cdot \frac{\partial g}{\partial u} - 2x \cdot \frac{\partial g}{\partial v}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} g(u,v) = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y} = 2y \frac{\partial g}{\partial u} + 2y \frac{\partial g}{\partial v}$$

$$x \frac{\partial f}{\partial y} - y \frac{\partial f}{\partial x} = 2xy \frac{\partial g}{\partial u} + 2xy \frac{\partial g}{\partial v} - 2xy \frac{\partial g}{\partial u} + 2xy \frac{\partial g}{\partial v} =$$

$$= 4xy \frac{\partial g}{\partial v} \overset{\text{vill}}{=} (x^3y + xy^3) g = xy(x^2+y^2) \cdot g \Rightarrow$$

$$\Rightarrow \left\{ \begin{matrix} x>0,\ y>0 \\ x^2+y^2 = u \end{matrix} \right\} \Rightarrow 4 \frac{\partial g}{\partial v} = u \cdot g \Leftrightarrow \frac{\partial g}{\partial v} - \frac{1}{4} u g = 0$$

$$\Rightarrow \left\{ \text{Integr. faktor: } e^{-\frac{1}{4}uv} \right\} \Rightarrow$$

$$\Rightarrow \frac{\partial}{\partial v}\left(e^{-\frac{uv}{4}} g(u,v) \right) = 0 \Rightarrow e^{-\frac{uv}{4}} g(u,v) = h(u)$$

$$\Leftrightarrow g(u,v) = e^{\frac{uv}{4}} h(u) \text{ för någon funktion } h$$

$$\Rightarrow f(x,y) = g(y^2+x^2, y^2-x^2) = e^{\frac{1}{4}(y^2+x^2)(y^2-x^2)} h(x^2+y^2)$$

Vill att: $f(x,x) = x^2 \Leftrightarrow h(2x^2) = x^2 \Leftrightarrow h(t) = \frac{t}{2}$

$$\therefore f(x,y) = \frac{1}{2}(x^2+y^2) e^{\frac{1}{4}(y^4-x^4)}$$

5. Låt $f(x,y,z) = y$, $g(x,y,z) = x^2+y^2+z^2-1$ och $h(x,y,z) = x+2y+2z$. Vi vill hitta extrempunkterna till f under bivillkoren $g=0$ och $h=0$.

$\Rightarrow L(x,y,z,\lambda,\mu) = y + \lambda(x^2+y^2+z^2-1) + \mu(x+2y+2z)$

$$\frac{\partial L}{\partial x} = 2x\lambda + \mu = 0 \iff \mu = -2x\lambda$$

$$\frac{\partial L}{\partial y} = 1 + 2y\lambda + 2\mu = 0 \quad (*)$$

$$\frac{\partial L}{\partial z} = 2z\lambda + 2\mu = 0 \Longrightarrow 2z\lambda - 4x\lambda = 0 \iff 2\lambda(z-2x) = 0$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2+z^2-1 = 0 \quad (\star)$$

$$\frac{\partial L}{\partial \mu} = x+2y+2z = 0 \quad (**)$$

$2\lambda(z-2x) = 0 \Rightarrow$ Två fall: I. $\lambda = 0$, II. $z = 2x$

I. $\lambda = 0 \Rightarrow \mu = -2x\lambda = 0 \Rightarrow \{\lambda = \mu = 0 \text{ i } (*)\} \Rightarrow$

$\Rightarrow 1 = 0$ ↯ Går ej!

II. $z = 2x$ i $(**)$: $x + 2y + 4x = 0 \iff y = -\frac{5x}{2}$

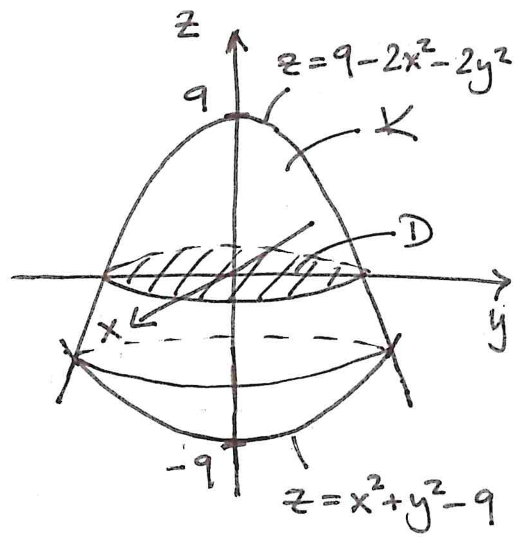
$$\overset{(\star)}{\Longrightarrow} x^2 + \frac{25}{4}x^2 + 4x^2 - 1 = 0 \iff x^2 = \frac{4}{45} \Rightarrow x = \pm\frac{2}{3\sqrt{5}}$$

$$\Rightarrow y = -\frac{5x}{2} = \mp\frac{5}{3\sqrt{5}}, \quad z = 2x = \pm\frac{4}{3\sqrt{5}}$$

$\therefore \frac{1}{3\sqrt{5}}(2,-5,4)$ ← Punkt med minst y-koordinat

$\frac{1}{3\sqrt{5}}(-2,5,-4)$ ← Punkt med störst y-koordinat

6. Totala volymen $= \iiint_K 1\,dV$



Cylindriska koord. lämpliga

Då ytorna skär varandra är radien:

$$9-2r^2 = r^2-9 \iff 18 = 3r^2$$

$$\iff r^2 = 6 \Rightarrow r = \sqrt{6}$$

$$\iiint_K 1\,dV = \left\{ \begin{array}{c} r^2-9 \le z \le 9-2r^2 \\ 0 \le r \le \sqrt{6} \\ 0 \le \theta \le 2\pi \end{array} \right\} = \int_0^{2\pi}\left(\int_0^{\sqrt{6}}\left(\int_{r^2-9}^{9-2r^2} r\,dz\right)dr\right)d\theta =$$

$$= 2\pi\int_0^{\sqrt{6}} r\left(9-2r^2-(r^2-9)\right)dr = 2\pi\int_0^{\sqrt{6}}(18r-3r^3)\,dr =$$

$$= 2\pi\left[9r^2-\frac{3}{4}r^4\right]_0^{\sqrt{6}} = 2\pi\left(54-\frac{3}{4}\cdot 36\right) = 54\pi$$

Volym över xy-planet $= \iint_D (9-2x^2-2y^2)\,dA$

Polära koord. lämpliga. I xy-planet är radien:

$$0 = 9-2r^2 \Rightarrow r = \frac{3}{\sqrt{2}}$$

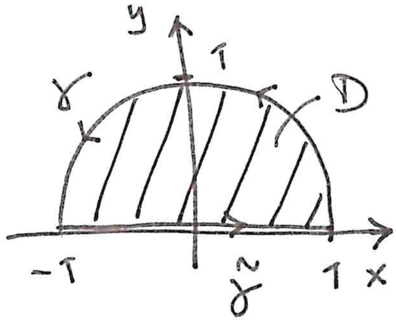
$$\iint_D (9-2x^2-2y^2)\,dA = \int_0^{2\pi}\left(\int_0^{3/\sqrt{2}}(9-2r^2)\,r\,dr\right)d\theta =$$

$$= \cancel{2}\pi\left[\frac{9r^2}{\cancel{2}}-\frac{1}{\cancel{2}}r^4\right]_0^{3/\sqrt{2}} = \pi\left(\frac{81}{2}-\frac{81}{4}\right) = \frac{81\pi}{4} = \frac{3^4\pi}{4}$$

$$\text{Andel} = \frac{3^4\pi}{4}\Big/54\pi = \frac{3^4\pi}{4}\cdot\frac{1}{2\cdot 3^3\pi} = \underline{\underline{\frac{3}{8}}}$$

7. Låt

$\mathbb{F}(x,y) = \left(2xy - x^2 + y^2\sin(xy^2),\ x + y^2 + 2xy\sin(xy^2)\right)$



Vill beräkna: $\int_\gamma \mathbb{F} \cdot d\mathbb{r}$

Vi kompletterar med linjestycket $\tilde{\gamma}$ enligt figuren och använder Greens sats:

$$\int_\gamma \mathbb{F}\cdot d\mathbb{r} + \int_{\tilde{\gamma}} \mathbb{F}\cdot d\mathbb{r} = \oint_{\gamma+\tilde{\gamma}} \mathbb{F}\cdot d\mathbb{r} = \left\{ \begin{array}{c} \text{Greens} \\ \text{sats} \end{array} \right\} =$$

$$= \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \underbrace{dxdy}_{dA} =$$

$$= \iint_D \left(1 + 2y\sin(xy^2) + 2xy^3\cos(xy^2) - \left(2x + 2y\sin(xy^2) + 2xy^3\cos(xy^2)\right)\right)$$

$$= \iint_D (1-2x)\,dA = \left\{ \begin{array}{c} \text{Polära} \\ \text{koord.} \end{array} \right\} = \int_0^\pi \left(\int_0^1 (1 - 2r\cos\theta)\, r\,dr \right) d\theta =$$

$$= \int_0^\pi \left[\frac{r^2}{2} - \frac{2}{3} r^3 \cos\theta \right]_0^1 d\theta = \int_0^\pi \left(\frac{1}{2} - \frac{2}{3}\cos\theta \right) d\theta =$$

$$= \left[\frac{1}{2}\theta - \frac{2}{3}\sin\theta \right]_0^\pi = \frac{\pi}{2} - \frac{2}{3}\sin(\pi) = \frac{\pi}{2}$$

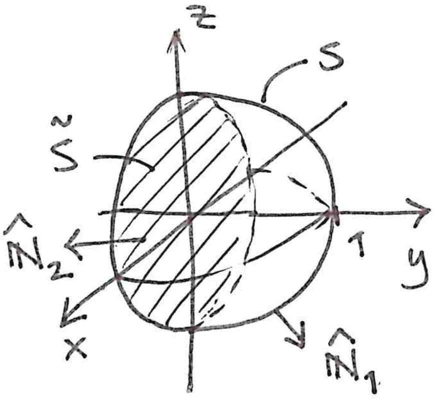
$$\int_{\tilde{\gamma}} \mathbb{F}\cdot d\mathbb{r} = \left\{ \begin{array}{l} \tilde{\gamma}: \mathbb{r}(t) = (t,0),\ -1 \le t \le 1 \\ \mathbb{r}'(t) = (1,0) \end{array} \right\} = \int_{-1}^{1} \mathbb{F}(t,0)\cdot(1,0)\,dt =$$

$$= \int_{-1}^{1} (-t^2, t)\cdot(1,0)\,dt = -\int_{-1}^{1} t^2\,dt = -\left[\frac{t^3}{3}\right]_{-1}^{1} = -\frac{2}{3}$$

$$\int_\gamma \mathbb{F}\cdot d\mathbb{r} = \oint_{\gamma+\tilde{\gamma}} \mathbb{F}\cdot d\mathbb{r} - \int_{\tilde{\gamma}} \mathbb{F}\cdot d\mathbb{r} = \frac{\pi}{2} - \left(-\frac{2}{3}\right) = \underline{\underline{\frac{\pi}{2} + \frac{2}{3}}}$$

8. Vill beräkna

$$\iint_S \mathbb{F}\cdot\hat{\mathbb{N}}_1\, dS$$



Komplettera med $\tilde{S}$ och använd Gauss sats

$$\iint_S \mathbb{F}\cdot\hat{\mathbb{N}}_1 dS + \iint_{\tilde{S}} \mathbb{F}\cdot\hat{\mathbb{N}}_2 dS = \oiint_{S+\tilde{S}} \mathbb{F}\cdot\hat{\mathbb{N}} dS = \left\{\begin{array}{c}\text{Gauss}\\ \text{sats}\end{array}\right\} =$$

$$= \iiint_K \nabla\cdot\mathbb{F}\, dV = \iiint_K (1+2(y-1)+1)\, dV = 2\iiint_K y\, dV =$$

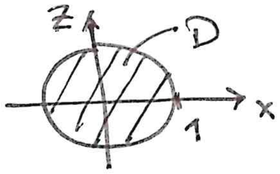
$$= \left\{\begin{array}{ll}\text{sfäriska} & 0\le R\le 1\\ \text{koord.} & 0\le\theta\le\pi\\ & 0\le\phi\le\pi\end{array}\right\} = 2\int_0^\pi\left(\int_0^\pi\left(\int_0^1 R\sin\phi\sin\theta\cdot R^2\sin\phi\, dR\right)d\phi\right)d\theta$$

$$= 2\int_0^\pi \sin\theta\, d\theta\cdot\int_0^\pi \frac{1-\cos(2\phi)}{2}\, d\phi\cdot\int_0^1 R^3 dR =$$

$$= 2\left[-\cos\theta\right]_0^\pi\cdot\left[\frac{1}{2}\phi-\frac{\sin(2\phi)}{4}\right]_0^\pi\cdot\left[\frac{R^4}{4}\right]_0^1 =$$

$$= 2\cdot 2\cdot\frac{\pi}{2}\cdot\frac{1}{4} = \frac{\pi}{2}$$

$$\iint_{\tilde{S}} \mathbb{F}\cdot\hat{\mathbb{N}}_2\, dS = \left\{\begin{array}{l}\mathbb{r}(x,z)=(x,0,z),\ (x,z)\in D\\ \hat{\mathbb{N}}_2 dS=(0,-1,0)\,dx\,dz\end{array}\right.$$



$$\} =$$

$$= \iint_D \mathbb{F}(x,0,z)\cdot(0,-1,0)\,dx\,dz = \iint_D (x,1,z)\cdot(0,-1,0)\,dx\,dz =$$

$$= -\iint_D 1\, dA = -\text{Area}(D) = -\pi$$

$$\iint_S \mathbb{F}\cdot\hat{\mathbb{N}}_1 dS = \iiint_K \nabla\cdot\mathbb{F}\, dV - \iint_{\tilde{S}} \mathbb{F}\cdot\hat{\mathbb{N}}_2 dS = \frac{\pi}{2}-(-\pi) = \underline{\underline{\frac{3\pi}{2}}}$$