VSM167 Finite element method - basics
Re-exam 2019-04-26, 14:00-18:00

Instructor: Martin Fagerström (phone 070-224 8731). The instructor will visit the exam around 15:00 and 16:30.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on Friday 10 May 2019 the latest.

Review: It will be possible to review the grading at the Division of Material and Computational Mechanics (floor 3 in M-building). Please make an appointment with Martin Fagerström if you wish to review the exam and/or discuss the grades.

Permissible aids: Chalmers type approved pocket calculator. **Note**: A formula sheet is appended to this exam thesis.

Problem 1

Consider a cylinder to be used in a shaft shrink-fit, see Figure 1a. To expand the cylinder such that it can be placed on the outside of the two shafts to be connected, the cylinder is heated by subjecting it to an internal heat flux $\overline{q}$ on the internal surface.

The heat is supplied in such a way that no temperature variations occur neither in the circumferential direction, nor in the longitudinal direction. This means that it is enough to consider a 1D heat flow problem (in the radial direction) to calculate the temperature distribution in the cylinder. For this case, the 1D heat flow is given by Fourier's law as $q(r) = -k(r)\frac{dT}{dr}$. Furthermore, the conditions on the outer surface of the cylinder is to be considered as convective, with the heat transfer coefficient α and external temperature T_{air}.

To derive the governing equations, it is helpful to consider a small section of the cylinder (given by a small circumferential angle $\Delta\phi$) at the radial position r, see Figure 1b. As no heat (except for that supplied to the inner surface of the cylinder) is added to the system, a simple heat balance gives at hand that:

$$q(r) \cdot A(r) = q(r + \Delta r) \cdot A(r + \Delta r) = \text{constant}, \text{ or } \frac{d}{dr}\left(q(r)A(r)\right) = 0$$

where $A(r)$ is the surface area at coordinate r equal to:

$$A(r) = \Delta\phi \cdot r \cdot L$$

where L is the length of the cylinder. Since this length is constant over r, and since not variations occur in the circumferential direction, the heat balance equation can be simplified to:

$$\frac{d}{dr}\left(r \cdot q(r)\right) = 0.$$

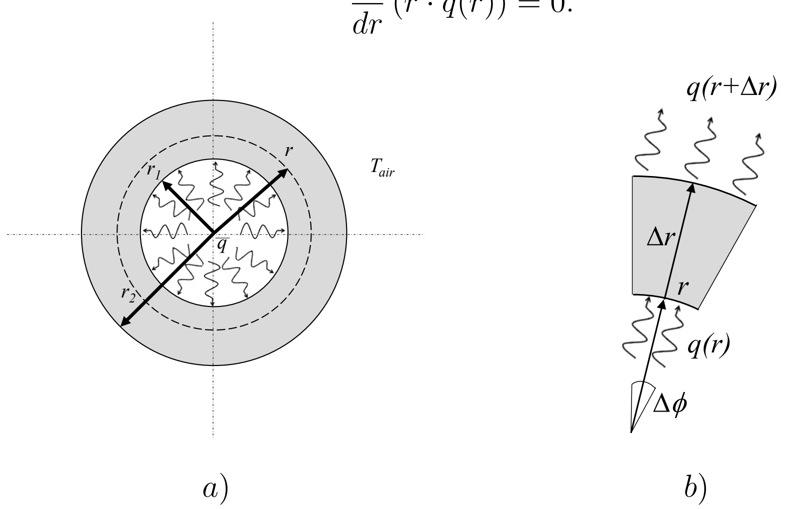


Figure 1: a) Cylinder to be considered in Problem 1 subjected to an internal heat $\overline{q}$. b) Close up of a small piece of the cylinder at radial coordinate r.

Tasks on the next page!

Tasks:

(a) By considering the 1D heat flow balance in the radial direction, **derive and state the strong form of the problem**. (0.5p)

(b) Given the strong form of the problem, **derive and state the full weak form of the problem at hand**. (1.0p)

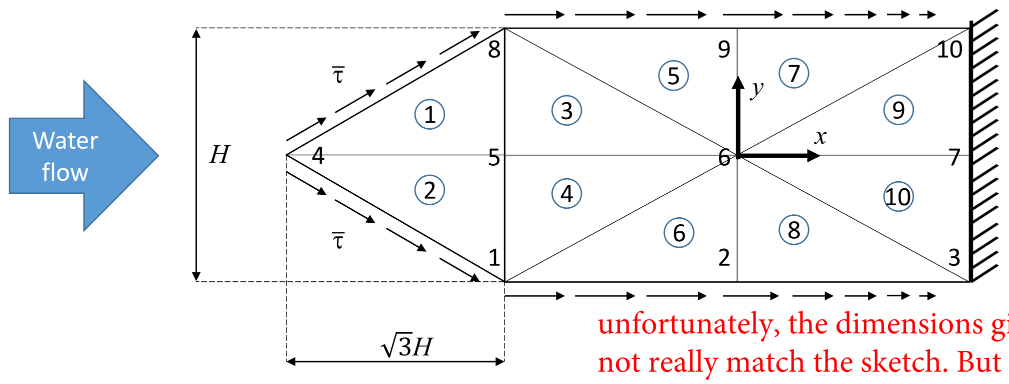
(c) Given the weak form of the problem, **derive and state the FE form of the problem at hand**. Be careful to explain the contents of any vectors or matrices you introduce. (0.5p)

(d) Consider specifically a problem with a constant heat conductivity $k(r) = k$, discretised with five 1D elements with linear shape functions. **Calculate the element stiffness (also denoted conductivity) matrix for the outer-most element** (the element with one node on the outer surface). (0.5p)

(e) For the same problem, **calculate any contribution to the global stiffness (also denoted conductivity) matrix associated with the boundary conditions**. (0.5p)

Problem 2

Consider the water divider as shown in Figure 2. As a result of the separation of water flow, shear tractions act on the outer surface as indicated in the figure (let us disregard any traction component normal to the surface). In particular, the shear traction are constant (with magnitude equal to $\overline{\tau}$) on the inclined surfaces (from node 4 to 8 and from node 4 to 1, respectively), and decreases from left to right along the planar edges (from node 8 to 10 and from node 1 to 3 respectively).



unfortunately, the dimensions given in the figure did not really match the sketch. But the solution is adapted after the values given

Figure 2: Water divider considered in Problem 2.

As no variations are considered perpendicular to the plane shown in Figure 2, the problem can be considered as a 2D plane strain problem. The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\boldsymbol{v}\right)^{\mathrm{T}} \boldsymbol{D}\tilde{\nabla}\boldsymbol{u}\, t\, \mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\boldsymbol{b}\, t\, \mathrm{d}A + \int_{\mathcal{L}_g} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{t}\, t\, \mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{h}\, t\, \mathrm{d}\mathcal{L}$$

where A denotes the area of the specimen, t its thickness, $\boldsymbol{v} = [v_x, v_y]^T$ a vector arbitrary weight function, $\boldsymbol{u} = [u_x, u_y]^T$ the displacement field (x- and y-component), $\mathcal{L}_g$ the part of the boundary with prescribed degrees of freedom ($\boldsymbol{g}$), $\mathcal{L}_h$ the part of the boundary with prescribed tractions ($\boldsymbol{h}$) and where $\boldsymbol{D}$ is the constitutive matrix relating stresses ($\boldsymbol{\sigma}$) and strains ($\boldsymbol{\varepsilon} = \tilde{\nabla}\boldsymbol{u}$) on Voigt form such that

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}.$$

As the problem is under the state of plane strain, the $\boldsymbol{D}-$matrix becomes

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}.$$

Task on the next page!

Tasks:

(a) Introduce suitable FE approximations for $\boldsymbol{v}$ and $\boldsymbol{u}$ and then **derive and state the FE formulation of the current problem**. Be careful to clearly indicate the contents of any matrices you introduce (or you will not be able to get full points for this subtask). (1.0p)

Please note that there is no need at this point to introduce the specific form of the traction boundary conditions.

(b) Do the following:

Define an appropriate numbering scheme for the degrees-of-freedom associated with the displacement field.

Define the topology matrix `Edof` (or similar) corresponding to your numbering scheme which links degrees-of-freedom to the element numbering. It is enough to write the first two lines of that matrix.

Finally, **write down**, with pen and paper, **the MATLAB code necessary to assemble the element stiffness matrix** (you may call it `Ke`) into the global stiffness matrix (`K`).

On your exam, you would be asked to actually write the code

(c) Consider specifically element 1 and the edge between nodes 4 and 8. For this edge, **define the traction vector expressed in the global coordinates that is acting on this edge**. Then **use this to calculate that traction contribution to the global load vector**. For full points, both the values and how these are assembled needs to be correctly explained. (1.0p)

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Problem 3

Please consider the 2D heat flow problem in Figure 3. The geometry is corresponding to Cooks membrane for which the upper and lower boundaries are to be considered as convective with heat transfer coefficient α. Furthermore, the left boundary is insulated and the right boundary is subjected to a time-varying external heat flux $q(\tau)$ adding heat to the body (see the figure). At time $\tau = 0$, the temperature of the membrane is uniform and equal to the outer temperature T_{out}.

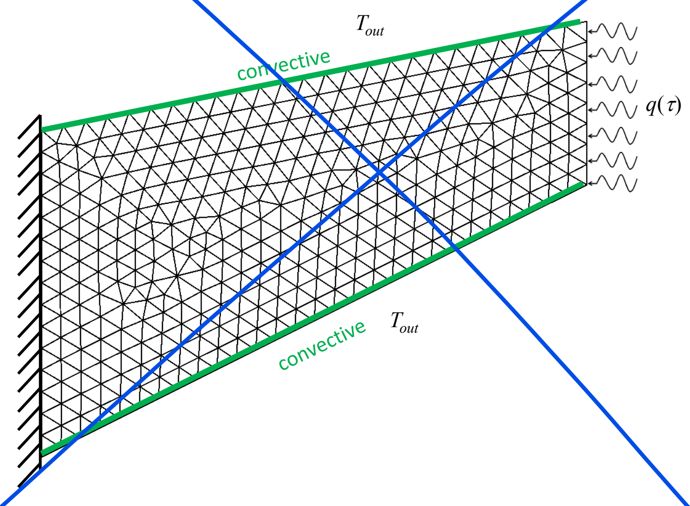


Figure 3: Illustration of the Cooks membrane problem for transient 2D heat flow (Problem 3)

Tasks:

(a) By considering the heat balance of a subdomain Ω' of the membrane, please **derive and state the strong form of the current 2D initial boundary value problem** to determine the temperature distribution history for $\tau \geq 0$. In the derivations, please also consider the possibility of having an external heat supply Q. (1.0p)

More tasks on the next page!

(b) The corresponding weak form of the governing equation for this particular problem (uniform thickness) is given by:

$$\int_A v\rho c\dot{T}\,\mathrm{d}A + \int_A (\nabla v)^T \boldsymbol{D}\nabla T\,\mathrm{d}A = -\int_{\mathcal{L}} vq_n\,\mathrm{d}\mathcal{L} + \int_A vQ\,\mathrm{d}A$$

where v is an arbitrary weight function, A is the domain of the membrane, ρ is the density of the material, c is the specific heat, $\dot{T}$ is the time derivative of the temperature, $\mathcal{L}$ is the outer boundary, $\boldsymbol{D}$ is the material conductivity matrix, ∇ is the 2D gradient operator, Q is the external heat supply and q_n is the heat outflux at the boundary.

Introduce suitable FE approximations for v and T and then **derive and state the semi-discrete FE formulation** (without introducing any time stepping scheme) of the initial boundary value problem. (1.0p)

(c) Under the assumption that the temperature varies linearly between two instants in time, τ_n and τ_{n+1} (with $\tau_{n+1} - \tau_n = \Delta\tau$) such that the degrees of freedom at time $\tau_{n+\theta} = \tau_n + \theta\Delta\tau$ is given by:

$$\boldsymbol{a}_{n+\theta} = \boldsymbol{a}_n + \theta\left(\boldsymbol{a}_{n+1} - \boldsymbol{a}_n\right) = (1-\theta)\,\boldsymbol{a}_n + \theta\boldsymbol{a}_{n+1},$$

please **derive the matrix equations on the form:**

$$\tilde{\boldsymbol{K}}(\theta)\boldsymbol{a}_{n+1} = \tilde{\boldsymbol{f}}(\theta)$$

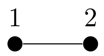
that can be used to calculate the temperature distribution at time $\boldsymbol{\tau} = \boldsymbol{\tau}_{n+1}$. (1.0p)

Hint: Please note that in this format $\tilde{\boldsymbol{f}}(\theta)$ will depend on boundary conditions and the external heat supply, but also on the values of the degrees-of-freedom from time step τ_n.

1 Shape functions

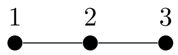
1.1 1D, linear

$$N_1^e = -\frac{1}{L}(x - x_2) \tag{1a}$$
$$N_2^e = \frac{1}{L}(x - x_1) \tag{1b}$$

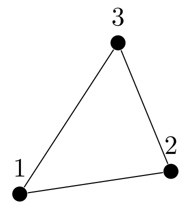


1.2 1D, quadratic

$$N_1^e = \frac{2}{L^2}(x - x_2)(x - x_3) \tag{2a}$$
$$N_2^e = -\frac{4}{L^2}(x - x_1)(x - x_3) \tag{2b}$$
$$N_3^e = \frac{2}{L^2}(x - x_1)(x - x_2) \tag{2c}$$



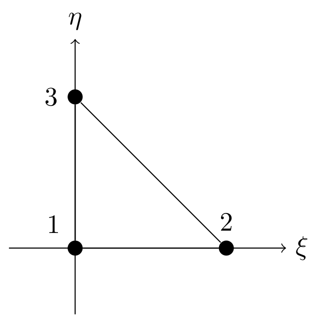
1.3 2D, linear triangle



$$N_1^e = \frac{1}{2A}(x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y) \tag{3a}$$
$$N_2^e = \frac{1}{2A}(x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y) \tag{3b}$$
$$N_3^e = \frac{1}{2A}(x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y) \tag{3c}$$

Parent element:

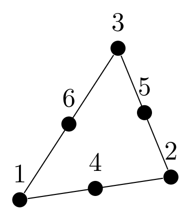
$$\overline{N}_1^e = 1 - \xi - \eta \tag{4a}$$
$$\overline{N}_2^e = \xi \tag{4b}$$
$$\overline{N}_3^e = \eta \tag{4c}$$

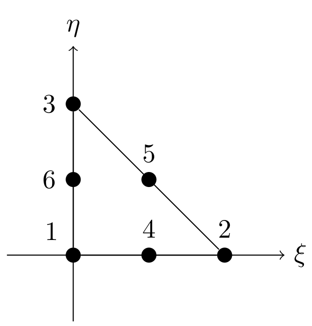


1.4 2D, Quadratic triangle

Parent element:

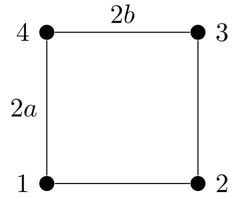
$$\overline{N}_1^e = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \tag{5a}$$
$$\overline{N}_2^e = \xi(2\xi - 1) \tag{5b}$$
$$\overline{N}_3^e = \eta(2\eta - 1) \tag{5c}$$
$$\overline{N}_4^e = 4\xi(1 - \xi - \eta) \tag{5d}$$
$$\overline{N}_5^e = 4\xi\eta \tag{5e}$$
$$\overline{N}_6^e = 4\eta(1 - \xi - \eta) \tag{5f}$$





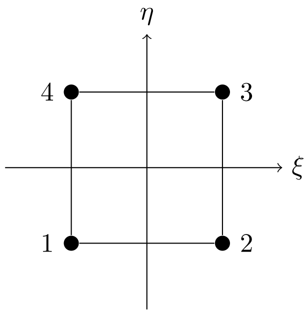
1.5 2D, bilinear

$$N_1^e = \frac{1}{4ab}(x - x_2)(y - y_4) \tag{6a}$$
$$N_2^e = -\frac{1}{4ab}(x - x_1)(y - y_3) \tag{6b}$$
$$N_3^e = \frac{1}{4ab}(x - x_4)(y - y_2) \tag{6c}$$
$$N_4^e = -\frac{1}{4ab}(x - x_3)(y - y_1) \tag{6d}$$



Parent element:

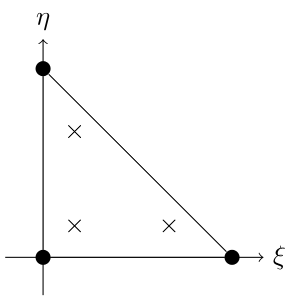
$$\overline{N}_1^e = \frac{1}{4}\,(\xi - 1)\,(\eta - 1) \tag{7a}$$
$$\overline{N}_2^e = -\frac{1}{4}\,(\xi + 1)\,(\eta - 1) \tag{7b}$$
$$\overline{N}_3^e = \frac{1}{4}\,(\xi + 1)\,(\eta + 1) \tag{7c}$$
$$\overline{N}_4^e = -\frac{1}{4}\,(\xi - 1)\,(\eta + 1) \tag{7d}$$



2 Gauss points

n	ξ_i	W_i
1	0.0000000000000000	2.0000000000000000
2	±0.5773502691896257	1.0000000000000000
3	0.0000000000000000	0.8888888888888889
	±0.7745966692414834	0.5555555555555556
4	±0.3399810435848563	0.6521451548625460
	±0.8611363115940525	0.3478548451374544

Table 1: Position of Gauss points ξ_i and corresponding weight W_i for n Gauss points.



n	(ξ_i, η_i)	W_i
1	$\left(\frac{1}{3}, \frac{1}{3}\right)$	$\frac{1}{2}$
3	$\left(\frac{1}{6}, \frac{1}{6}\right)$	$\frac{1}{6}$
	$\left(\frac{2}{3}, \frac{1}{3}\right)$	$\frac{1}{6}$
	$\left(\frac{1}{6}, \frac{2}{3}\right)$	$\frac{1}{6}$

3 Green-Gauss theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$.

$$\int_A \phi \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{w}\,\mathrm{d}A + \int_A (\boldsymbol{\nabla}\phi)^{\mathrm{T}} \boldsymbol{w}\,\mathrm{d}A = \int_{\mathcal{L}} \boldsymbol{n}^{\mathrm{T}}(\phi \boldsymbol{w})\,\mathrm{d}\mathcal{L} \tag{8}$$

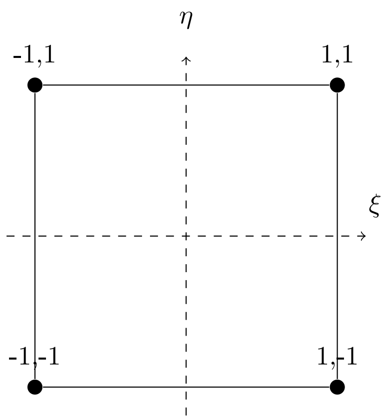
4 Gauss divergence theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$, $\mathrm{div}(\boldsymbol{w}) = \nabla^{\mathrm{T}}\boldsymbol{w}$.

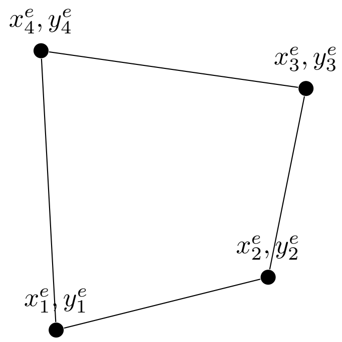
$$\int_A \boldsymbol{\nabla}^{\mathrm{T}}(\phi \boldsymbol{w})\, \mathrm{d}A = \int_{\mathcal{L}} (\phi \boldsymbol{w})^{\mathrm{T}}\, \boldsymbol{n}\, \mathrm{d}\mathcal{L}$$

5 Isoparametric mapping

Parent domain



Global domain



$$\mathbf{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}, \mathbf{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

$$x = x(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\, \mathbf{x}^e \tag{9}$$

$$y = y(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\, \mathbf{y}^e \tag{10}$$

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial x} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial y} \end{bmatrix} = \left(\mathbf{J}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \xi} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \eta} \end{bmatrix} \tag{12}$$

6 Matrix inversion

The inverse of the matrix $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is given by:

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \quad \text{with } \det(\mathbf{M}) = M_{11}M_{22} - M_{12}M_{21}. \tag{13}$$

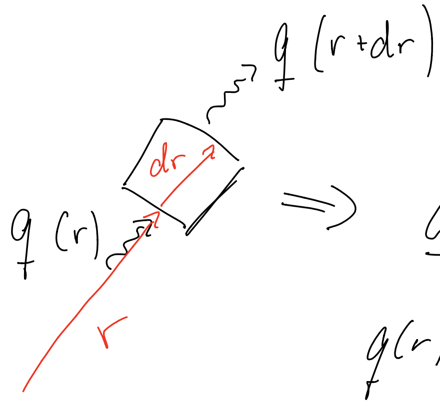
7 Stresses and strains

Hooke's generalised law: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

2D Strain-displ. relation: $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{bmatrix} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u},\ \boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix},\ \tilde{\boldsymbol{\nabla}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$

Solutions

P1 a)



$\Longrightarrow$ Given $\frac{d}{dr}\left(r \cdot q(r)\right) = 0$

$$q(r) = -k\frac{dT}{dr}$$

Insert expression for $q(r)$ in diff eq: $\Rightarrow$

$$\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

Add boundary conditions:

$$q_n(r_1) = -q(r_1) = -\bar{q}$$

$$q_n(r_2) = \alpha\left(T(r_2) - T_{air}\right)$$

Together they give the strong form as:

$$\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

$$q_n(r_1) = -\bar{q}$$

$$q_n(r_2) = \alpha\left[T(r_2) - T_{air}\right]$$

P1b) Multiply the strong form equation with arbitrary weight function $v(r)$ & integrate over the domain:

$$\int_{r_1}^{r_2} v \frac{d}{dr}\left(kr\frac{dT}{dr}\right) = \left[v \cdot r \cdot k \frac{dT}{dr}\right]_{r_1}^{r_2} - \int_{r_1}^{r_2} \frac{dv}{dr} k \cdot r \frac{dT}{dr}\, dr$$

Inserting the BC's yields

$$\int_{r_1}^{r_2} \frac{dv}{dr} k \cdot r \frac{dT}{dr} = -v(r_2) \cdot r_2 \cdot \alpha\left(T(r_2) - T_{air}\right) + v(r_1) \cdot r_1 \cdot \bar{q}$$

P1c) To obtain the FE form, we insert approximations of T & v as

$$T \approx T_h = \sum_{i=1}^{n} N_i(r)\, T_i = \mathbf{N}\mathbf{a} \quad \text{with}$$

$$\mathbf{N} = [N_1 \; N_2 \; \dots \; N_n] \;\&\; \mathbf{a} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}$$

$$\Rightarrow \frac{dT}{dr} = \frac{d\mathbb{N}}{dr}\,\mathbb{a} = \mathbb{B}\,\mathbb{a} \quad \text{with}$$

$$\mathbb{B} = \left[\frac{dN_1}{dr} \quad \frac{dN_2}{dr} \quad \cdots \quad \frac{dN_n}{dr} \right]$$

Using Galerkin method, we also obtain:

$$v \approx v_h = \mathbb{N}\,\mathbb{c} \quad \text{with} \quad \mathbb{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \text{ (arbitrary)}$$

$$\Rightarrow \frac{dv}{dr} = \mathbb{B}\,\mathbb{c} = \mathbb{c}^T \mathbb{B}^T$$

Insert approximations in the weak form:

$$\mathbb{c}^T \left[\underbrace{\int_{r_1}^{r_2} \mathbb{B}^T k \cdot r\, \mathbb{B}\, dr}_{\mathbb{K}}\, \mathbb{a} + \underbrace{\alpha\, \mathbb{N}^T(r_2) \cdot r_2\, \mathbb{N}(r_2)}_{\mathbb{K}_c}\, \mathbb{a} \underbrace{- \alpha\, \mathbb{N}^T(r_2) \cdot r_2 \cdot T_{air} - \mathbb{N}^T(r_1) \cdot r_1 \cdot \bar{q}}_{\mathbb{f}_b} \right] = 0$$

$$\Rightarrow (\mathbb{K} + \mathbb{K}_c)\,\mathbb{a} = \mathbb{f}_b$$

P1 d) Discretise the radial direction with (n-1) elements with equal length

$$L_e = \frac{r_2 - r_1}{(n-1)}$$

The outermost element stiffness matrix is then obtained from

$$\mathbb{K}^e = \int_{r_2-L_e}^{r_2} \mathbb{B}^{eT} k \cdot r \, \mathbb{B}^e \, dr$$

As the element shape functions will go from 1 to 0 (and vice versa) over the element length L_e, the element shape function derivative matrix (vector) becomes

$\mathbb{B}^e = \left[-\frac{1}{L_e} \quad \frac{1}{L_e} \right]$ whereby:

$$\mathbb{K}^e = \int_{r_2-L_e}^{r_2} \begin{bmatrix} -\frac{1}{L_e} \\ \frac{1}{L_e} \end{bmatrix} k \cdot r \left[-\frac{1}{L_e} \quad \frac{1}{L_e} \right] dr =$$

$$= k \begin{bmatrix} \frac{1}{L_e^2} & -\frac{1}{L_e^2} \\ -\frac{1}{L_e^2} & \frac{1}{L_e^2} \end{bmatrix} \int_{r_2 - L_e}^{r_2} r\,dr$$

$$= k \begin{bmatrix} \frac{1}{L_e^2} & -\frac{1}{L_e^2} \\ -\frac{1}{L_e^2} & \frac{1}{L_e^2} \end{bmatrix} \left[r^2 \right]_{r_2 - L_e}^{r_2}$$

$$= k\left[r_2^2 - \left(r_2^2 - 2 r_2 L_e + L_e^2 \right) \right] \begin{bmatrix} \frac{1}{L_e^2} & -\frac{1}{L_e^2} \\ -\frac{1}{L_e^2} & \frac{1}{L_e^2} \end{bmatrix}$$

$$= k \left(2 r_2 L_e - L_e^2 \right) \begin{bmatrix} \frac{1}{L_e^2} & -\frac{1}{L_e^2} \\ -\frac{1}{L_e^2} & \frac{1}{L_e^2} \end{bmatrix} \quad ///$$

Ple) Contributions to the global stiffness comes from the convective BC's via $\mathbb{K}_c$

$$\mathbb{K}_c = \alpha r_2 \mathbb{N}^T(r_2)\, \mathbb{N}(r_2) =$$

$$= \alpha r_2 \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix}$$

///

P2/ $\mathbf{u} \simeq \mathbf{u}_h = \sum_{i=1}^{n} N_i \mathbf{a}_i^e = \mathbb{N}\mathbf{a}$ with

$$\underbrace{\mathbf{a}_i}_{\begin{bmatrix} u_{x,i} \\ u_{y,i} \end{bmatrix}}$$

$$\mathbb{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix}$$

where $N_i(x,y)$ is a 2D shape function associated with node i

$$\& \quad \mathbf{a} = \begin{bmatrix} u_{x,1} \\ u_{y,1} \\ u_{x,2} \\ u_{y,2} \\ \vdots \\ u_{x,n} \\ u_{y,n} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$\boldsymbol{\varepsilon} = \tilde{\nabla}\mathbf{u} \simeq \tilde{\nabla}\mathbf{u}_h = \tilde{\nabla}(\mathbb{N}\mathbf{a}) = \underbrace{(\tilde{\nabla}\mathbb{N})}_{\mathbb{B}}\mathbf{a} = \mathbb{B}\mathbf{a} \text{ with}$$

$$\mathbb{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & \cdots & \frac{\partial N_n}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & & 0 & \frac{\partial N_n}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & & \frac{\partial N_n}{\partial y} & \frac{\partial N_n}{\partial x} \end{bmatrix}$$

Using Galerkins method to approxi $\mathbb{v}$ we obtain:

$$\mathbb{v} \approx \mathbb{v}_h = \mathbb{N}\mathbb{c} \Rightarrow \mathbb{v}^T = \mathbb{c}^T \mathbb{N}^T$$

$$\tilde{\nabla}\mathbb{v} \approx \tilde{\nabla}\mathbb{v}_h = \mathbb{B}\mathbb{c} \Rightarrow (\tilde{\nabla}\mathbb{v}_h)^T = \mathbb{c}^T \mathbb{B}^T$$

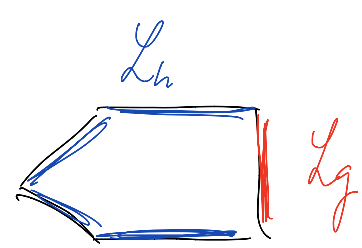
where $\mathbb{c}$ contains arbitrary coefficients.

Inserted in the weak form yields:

$$\mathbb{c}^T\left[\underbrace{\int_A \mathbb{B}^T \mathbb{D}\mathbb{B}\, t\, dA}_{\mathbb{K}}\,\mathbb{a} - \underbrace{\int_A \mathbb{N}^T \mathbb{b}\, t\, dA}_{\mathbb{f}_\ell} - \underbrace{\int_{\mathcal{L}_g} \mathbb{N}^T \mathbb{t}\, t\, d\mathcal{L}}_{\mathbb{f}_b^g} - \underbrace{\int_{\mathcal{L}_h} \mathbb{N}^T \mathbb{h}\, t\, d\mathcal{L}}_{\mathbb{f}_b^h}\right] = 0$$

$\mathbb{c}$ – arbitrary yields

$$\begin{cases} \mathbb{K}\mathbb{a} = \mathbb{f}_\ell + \mathbb{f}_b^g + \mathbb{f}_b^h \\ \mathbb{u} = 0 \text{ on } \mathcal{L}_g \\ \mathbb{t} = \mathbb{h} \text{ on } \mathcal{L}_h \end{cases}$$ with



P2b) To number the degrees of freedom we use that for node k then

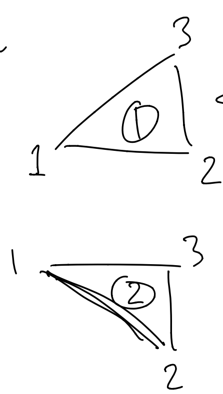
$$u_{x,k} = a_{2k-1}$$

$$u_{y,k} = a_{2k}$$

A possible form of Edof is

$$\begin{array}{l} \quad \text{el no} \\ \text{Edof} = \begin{bmatrix} 1 & 7 & 8 & 9 & 10 & 15 & 16 \\ 2 & 7 & 8 & 1 & 2 & 9 & 10 \\ \vdots & & & & & & \\ \vdots & & & & & & \end{bmatrix} \end{array}$$

with

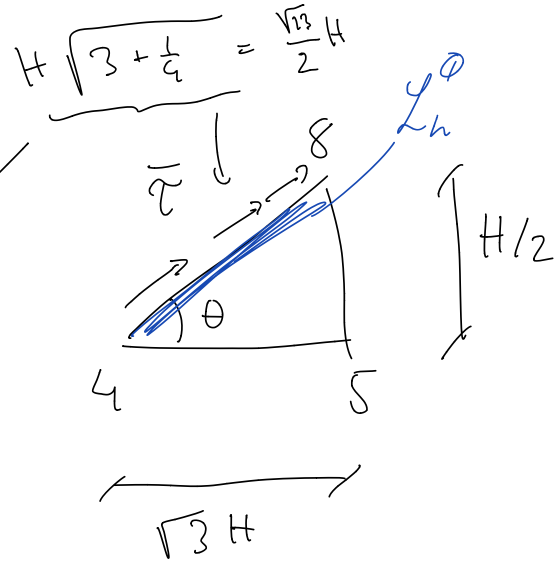


local node numbering

Given the element stiffness matrix Ke for element i, it can be assembled into the global stiffness matrix as:

$$K(Edof(i,2{:}end), Edof(i,2{:}end)) =$$
$$K(Edof(i,2{:}end), Edof(i,2{:}end)) + Ke$$

P2c)



$\tan\theta = \frac{1}{2\sqrt{3}} \Rightarrow$

$\sin\theta = \frac{1}{\sqrt{13}}$ (if following the probl. descr.)

$\cos\theta = \sqrt{\frac{3}{13}}\cdot 2$

along the edge between nodes 4 & 8 we have that $t = \begin{bmatrix} \bar{\tau}\cos\theta \\ \bar{\tau}\sin\theta \end{bmatrix} = \begin{bmatrix} \bar{\tau}\sqrt{\frac{12}{13}} \\ \bar{\tau}\frac{1}{\sqrt{13}} \end{bmatrix}$

To calculate the element boundary load vector we can use:

$$f_b^{(1)} = \int_{\mathcal{L}_h^{①}} \mathbb{N}^{e^T} \mathbb{t}\, t \, d\mathcal{L}$$

$$= \int_{\mathcal{L}_h^{①}} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ 0 & 0 \\ 0 & 0 \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix} \begin{bmatrix} \bar{\tau}\sqrt{\frac{12}{13}} \\ \bar{\tau}\frac{1}{\sqrt{13}} \end{bmatrix} t \, d\mathcal{L} =$$

$= ($ everything is constant $|\mathcal{L}_h^{①}| = \frac{\sqrt{13}\,H}{2}$

$$\sqrt{13}\,H\frac{t}{4} \begin{bmatrix} \sqrt{\frac{12}{13}}\bar{\tau} \\ \bar{\tau}\frac{1}{\sqrt{13}} \\ 0 \\ 0 \\ \sqrt{\frac{12}{13}}\bar{\tau} \\ \bar{\tau}\frac{1}{\sqrt{13}} \end{bmatrix} = \frac{Ht}{4} \begin{bmatrix} 2\sqrt{3}\,\bar{\tau} \\ \bar{\tau} \\ 0 \\ 0 \\ 2\sqrt{3}\,\bar{\tau} \\ \bar{\tau} \end{bmatrix} \begin{matrix} \leftarrow \text{added to pos } 7 \\ \leftarrow \text{——}\,''\,\text{——}\ 8 \\ \\ \\ \leftarrow \text{added to pos } 15 \\ \leftarrow \text{——}\,''\,\text{——}\ 16 \end{matrix}$$

P3a)

We start by introducing the internal energy e per unit weight which is a measure of the stored heat

The rate of change of e only assumed to depend on the temperature T becomes

$$\frac{de}{d\tau} = \underbrace{\frac{de}{dT}}_{c} \frac{dT}{d\tau} = c\dot{T}$$

where we introduced the specific heat $c = \frac{de}{dT}$

A simple heat balance of a subregion Ω' then says that the rate of change of the stored heat must equal heat inflow minus heat outflow:

$$\int_{\Omega'} \rho \dot{e} t \, d\Omega = \int_{\Omega'} Q t \, d\Omega - \int_{\partial\Omega'} q_n t \, d\Sigma$$

$\uparrow$ boundary of Ω'

$$\Rightarrow \int_{\Omega'} \rho c \dot{T} t \, d\Omega = \int_{\Omega'} Q t \, d\Omega - \int_{\partial\Omega'} q_n t \, d\Sigma$$

To end at the strong form, we need to rewrite the last term as:

$$\int_{\partial\Omega'} q_n t \, d\Sigma = \int_{\Omega'} \operatorname{div}(t \underline{q}) \, d\Sigma$$

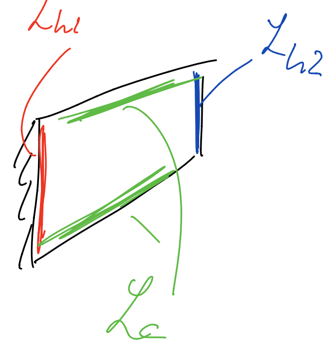
$$\Rightarrow \int_{\Omega'} \rho c \dot{T} t \, d\Omega = \int_{\Omega'} Q t \, d\Omega - \int_{\Omega'} \operatorname{div}(t \underline{q}) \, d\Sigma$$

which should hold for an arbitrary Ω'

meaning that

Strong form

$$\begin{cases} \rho c \dot{T} t + \mathrm{div}(t\,\mathbf{q}) = Q t & \text{for } \tau \leq 0 \\ \text{Adding BC:s} & \\ q_n = \mathbf{q}^T \mathbf{n} = 0 & \text{along } \mathcal{L}_{h1} \\ q_n = \alpha (T - T_{out}) & \text{along } \mathcal{L}_c \\ q_n = -q(\tau) & \text{along } \mathcal{L}_{h2} \\ \text{Adding initial condition:} & \\ T(x, y, \tau = 0) = T_{out} & \end{cases}$$



P3b/ The corresponding weak form equation is

$$\int_A v\rho c \dot{T}\, dA + \int_A (\nabla v)^T \mathbb{D}\, \nabla T\, dA = -\int_{\mathcal{L}} v q_n\, d\mathcal{L} + \int_A v Q\, dA$$

Introduce approximation for T

$T \simeq T_h = \mathbb{N}\mathbf{a}$ with $\mathbb{N} = [N_1(x,y)\; N_2(x,y) \ldots N_n(x,y)]$

$\dot{T} = \mathbb{N}\dot{\mathbf{a}}$

$$\mathbf{a} = \begin{bmatrix} T_1 \\ T_2 \\ \vdots \\ T_n \end{bmatrix}$$

$\nabla T \simeq \mathbb{B}\mathbf{a}$ with

$$\mathbb{B} = \nabla \mathbb{N} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \cdots & \frac{\partial N_n}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \cdots & \frac{\partial N_n}{\partial y} \end{bmatrix}$$

Using Galerkin method we also get that

$$v = \mathbb{N}c = c^T \mathbb{N}^T$$

& $\nabla v = \mathbb{B}c \Rightarrow (\nabla v)^T = c^T \mathbb{B}^T$

Insert approximations in the weak form:

$$c^T \left[\int_A \rho c\, \mathbb{N}^T \mathbb{N}\, dA\, \dot{a} + \int_A \mathbb{B}^T \mathbb{D} \mathbb{B}\, dA\, a + \int_{\mathcal{L}} \mathbb{N}^T q_n\, d\mathcal{L} - \int_A \mathbb{N}^T Q\, dA \right] = 0$$

c - arbitrary ⇒

$$\int_A \rho c\, \mathbb{N}^T \mathbb{N}\, dA\, \dot{a} + \int_A \mathbb{B}^T \mathbb{D} \mathbb{B}\, dA\, a + \int_{\mathcal{L}_{h1}} \mathbb{N}^T \cdot 0\, d\mathcal{L} + \int_{\mathcal{L}_c} \alpha \mathbb{N}^T (\mathbb{N} a - T_{out})\, d\mathcal{L}$$

$$- \int_{\mathcal{L}_{h2}} \mathbb{N}^T q(\tau)\, d\mathcal{L} - \int_A \mathbb{N}^T Q\, dA = 0$$

$$\Rightarrow \underbrace{\int_A \rho c\, \mathbb{N}^T \mathbb{N}\, dA}_{\mathbb{C}_1}\, \dot{a} + \underbrace{\int_A \mathbb{B}^T \mathbb{D} \mathbb{B}\, dA}_{\mathbb{K}}\, a + \overbrace{\int_{\mathcal{L}_c} \alpha \mathbb{N}^T \mathbb{N}\, d\mathcal{L}}^{\mathbb{K}_c}\, a =$$

$$\underbrace{\int_{\mathcal{L}_{h2}} \mathbb{N}^T q(\tau)\, d\mathcal{L}}_{\mathbb{f}_b} + \underbrace{\int_A \mathbb{N}^T Q\, dA}_{\mathbb{f}_l}$$

$$\mathbb{C}_1 \dot{a} + (\mathbb{K} + \mathbb{K}_c)\, a = \mathbb{f}_b + \mathbb{f}_l \qquad (\star)$$

P3c/

$$(\star\star)\begin{cases} a_{n+\theta} = (1-\theta) a_n + \theta a_{n+1} \\ \dot{a}_{n+\theta} = \frac{1}{\Delta\tau}(a_{n+1} - a_n) \\ \mathbb{f}_{n+\theta} = (1-\theta)\mathbb{f}_n + \theta \mathbb{f}_{n+1} \end{cases}$$

(or more accurately $\mathbb{f}_{n+\theta} = \mathbb{f}(\tau_\theta)$)

shown at the lecture

insert $\star\star$ in $\star$

$$\frac{1}{\Delta\tau}\mathbb{C}(a_{n+1} - a_n) + (\mathbb{K} + \mathbb{K}_c)\big((1-\theta)a_n + \theta a_{n+1}\big) = \mathbb{f}_b(\tau = \tau_\theta) + \mathbb{f}_l(\tau = \tau_\theta)$$

$$\underbrace{\left[\mathbb{C} + \Delta t\,\theta \left(\mathbb{K} + \mathbb{K}_c \right) \right]}_{\tilde{\mathbb{K}}(\theta)} a_{n+1} = \underbrace{\left[\mathbb{C} - \Delta t (1-\theta) (\mathbb{K} + \mathbb{K}_c) \right] a_n + \Delta \tau\, \mathbb{f}_0 (\tau_\theta) + \Delta \tau\, \mathbb{f}_1 (\tau_\theta)}_{\tilde{\mathbb{f}}(\theta)}$$