VSM167 Finite element method - basics
Exam 2019-01-18, 8:30-12:30

Instructor: Martin Fagerström (phone 070-224 8731). The instructor will visit the exam around 9:30 and 11:30.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on Friday 1 February 2019 the latest.

Review: It will be possible to review the grading at the Division of Material and Computational Mechanics (floor 3 in M-building). Please make an appointment with Martin Fagerström if you wish to review the exam and/or discuss the grades.

Permissible aids: Chalmers type approved pocket calculator. **Note**: A formula sheet is appended to this exam thesis.

Not for MHA021

Problem 1

Consider a wall as in Figure 1 (next page) with an outer layer of concrete (thickness h_1), a middle layer of mineral wool (thickness h_2) and an internal layer of plaster (thickness h_3). The material properties of the concrete, the mineral wool and the plaster are:

Thermal conductivity: k_1, k_2 and k_3;
Density: ρ_1, ρ_2 and ρ_3; and
Specific heat capacity: c_1, c_2 and c_3, respectively.

At the inner and outer surfaces, the heat transfer coefficients are α_{out} and α_{in}, respectively.

It is of design interest to calculate the wall temperature distribution (and its variation in time), especially the temperature values at the material interfaces as well as on the outside and on the inside of the wall, i.e. $T_1(\tau) - T_4(\tau)$ (see the figure).

In the initial steady state condition, the inner air temperature is $T = T_{in}$, and the outside air temperature is $T = T_{out}$. The solution of the steady state is such that the temperature variation through the wall is $T(x, \tau = 0) = T^0(x)$ yielding the temperatures $T_1 - T_4$ as:

$$\left\{ \begin{array}{c} T_1(\tau = 0) \\ T_2(\tau = 0) \\ T_3(\tau = 0) \\ T_4(\tau = 0) \end{array} \right\} = \left\{ \begin{array}{c} T_1^0 \\ T_2^0 \\ T_3^0 \\ T_4^0 \end{array} \right\}.$$

In the transient case, there is a sudden drop of the *outside air temperature* down to $T = \overline{T}$. For the analysis, it is considered sufficient to assume that this change in temperature occurs instantaneously at the time $\tau = 0$.

A simple heat balance including the ability for the material to store heat reveals that:

$$\rho c \dot{T} = -\frac{d}{dx}(Aq) + Q$$

where c is the specific heat capacity, $\dot{T} = dT/d\tau$ is rate of temperature increase, A is the wall cross section area (can be assumed to be unity for simplicity), q is the heat flux and Q is the external heat supply. Moreover, the heat flux is considered to obey Fourier's law such that

$$q = -k(x)\frac{dT}{dx}.$$

Figure and tasks on the next page

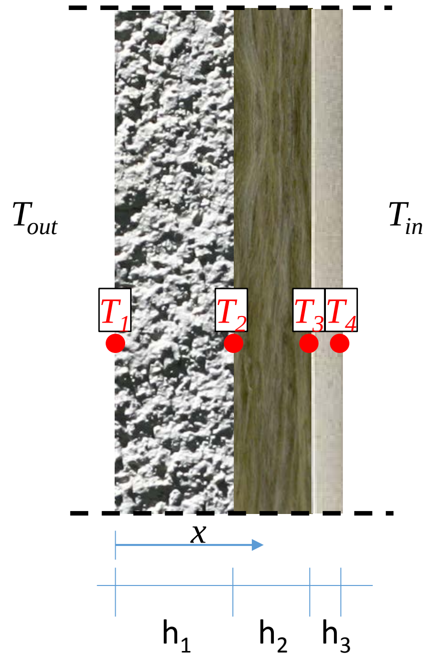


Figure 1: Geometry of the studied wall.

Tasks:

(a) Based on the general heat balance above, **derive and state the weak formulation of the current transient problem** from $\tau = 0$ and onwards. (**1.0p**)

(b) Using Galerkin's method, **derive and state the semi-discrete FE-formulation** for the transient case where the wall is discretised into N elements along x. Please note that you do *not* need to introduce any time integration scheme (such as the Generalized Midpoint Rule) for this task. However, be specific on the contents of any matrices you introduce (but you do not need to get to the level of detail of expressions for individual shape functions and their derivatives). (**0.5p**)

(c) How many element with linear approximation of the temperature field (through the thickness) would be needed to find the solution $T_1^0 - T_4^0$ in the stationary case? Clearly motivate your answer (if the answer is correct but the motivation is wrong or missing there will be no points awarded) (**0.5p**)

(d) Does the number of elements required for an accurate solution change if you are to consider the transient case with a sudden drop of the outside temperature? Motivate why or why not. Again, a clear motivation is needed to give points. (**0.5p**)

(e) Based on your FE-formulation derived in Task (b), **calculate the element capacity matrix**, often denoted $\boldsymbol{C}^e$, **for an element with length $L_e = h_2/4$ in the mineral wool section of the wall.** (**0.5p**)

Problem 2

Consider a concrete hot-water pipeline with an internal cylindrical hole for the water, cf. the rectangular cross-section sketch in Figure 2 (right). It is assumed that the puipeline has reached a steady state temperature distribution and that no heat is transferred along the pipeline whereby a simplification to a 2D stationary heat flow problem can be made.

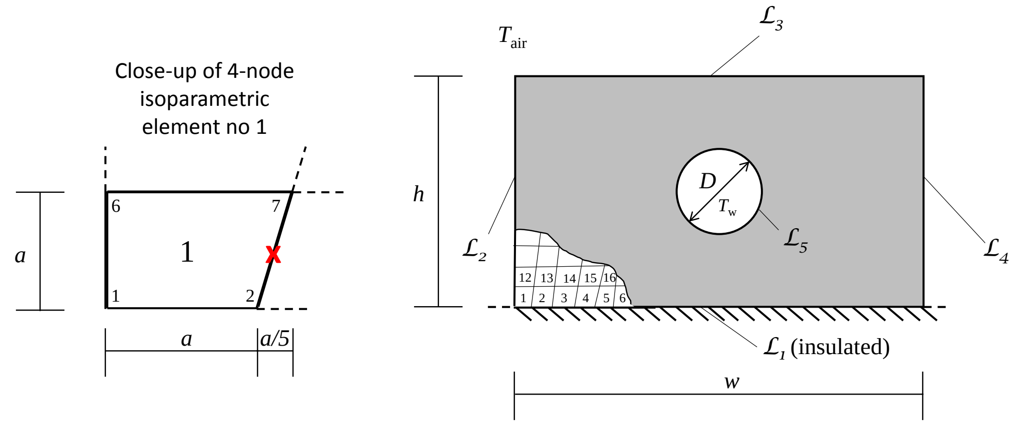


Figure 2: (left): Close-up of element no 1 analysed in subtasks (b) and (c). (right): Analysed cross-section of the hot water pipeline, indicating also parts of the FE mesh (in the lower left part).

For the whole cross-section, the boundary value problem for $T(x, y)$ on strong form becomes

$$
\begin{aligned}
\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{q} &= 0 \quad \text{in } \Omega, \\
q_n &= 0 \quad \text{on } \mathcal{L}_1, \\
q_n &= \alpha[T - T_{\text{air}}] \quad \text{on } \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4, \\
q_n &= \alpha[T - T_{\text{w}}] \quad \text{on } \mathcal{L}_5.
\end{aligned}
$$

The concrete is assumed to be isotropic (w.r.t heat flow) and obey Fourier's law $\boldsymbol{q} = -k\boldsymbol{\nabla}T$ with the thermal conductivity, k, assumed to be constant. For simplicity, the transfer coefficient α is assumed to be the same along all boundaries.

For the analysis, the cross-section is discretised in terms of 4-node isoparametric bilinear elements (as indicated in the lower left corner of the figure). In subtasks (b) and (c) below, one of these elements (element no 1) will be analysed more in detail. As shown in cf. Figure 2 (left), it has the nodal coordinates:

$$(x_1, y_1) = (0, 0), \quad (x_2, y_2) = (a, 0), \quad (x_6, y_6) = (0, a), \quad (x_7, y_7) = (6a/5, a).$$

The corresponding parent element occupies (as always) the domain $-1 \le \xi \le 1$, $-1 \le \eta \le 1$ in the local coordinates (ξ, η). **Tasks on the next page.**

(a) Determine the smallest part of the cross section that can be analysed (with respect to symmetry conditions). For this domain, **derive the weak form corresponding to the strong form above in the general format without suppressing boundary reactions**. (For simplicity, heat flow in a "slice" of thickness $t = 1$ (m) can be considered). Note that you may have to make an additional sketch of the domain that is analysed (in order to clarify any boundary condition statements). **(1.0p)**

(b) For the lower left element (element no 1), **compute the Jacobian matrix and its determinant associated with the isoparametric mapping in the midpoint of the element edge between nodes 2 and 6** (marked with a red X). **(1.0p)**

(c) Given a solution of the temperature field in the nodes according to

$$
\begin{aligned}
T_1 &= 3\ ^\circ\mathrm{C} \\
T_2 &= 4\ ^\circ\mathrm{C} \\
T_6 &= 2\ ^\circ\mathrm{C} \text{ and} \\
T_7 &= 5\ ^\circ\mathrm{C}
\end{aligned}
$$

set up the expression to compute the gradient of the temperature ∇T in the same point as in Task (b) (marked with a red X in the figure). Please note that if your expression involves matrix multiplications, each matrix component needs to be defined but there is no need to actually perform the matrix multiplications. **(1.0p)**

Problem 3

The Cook membrane problem is a classical 2D test case for linear static analysis named after the author, R. D. Cook who first reported it in 1974. The structure consist of a trapezoidal surface in the $x-y$ plane (see figure). The structure is clamped along the left edge and it is loaded by a edge shear traction load t_y along the right edge.

In the figure below (right) there is a close up of the element discretisation at the upper left corner.

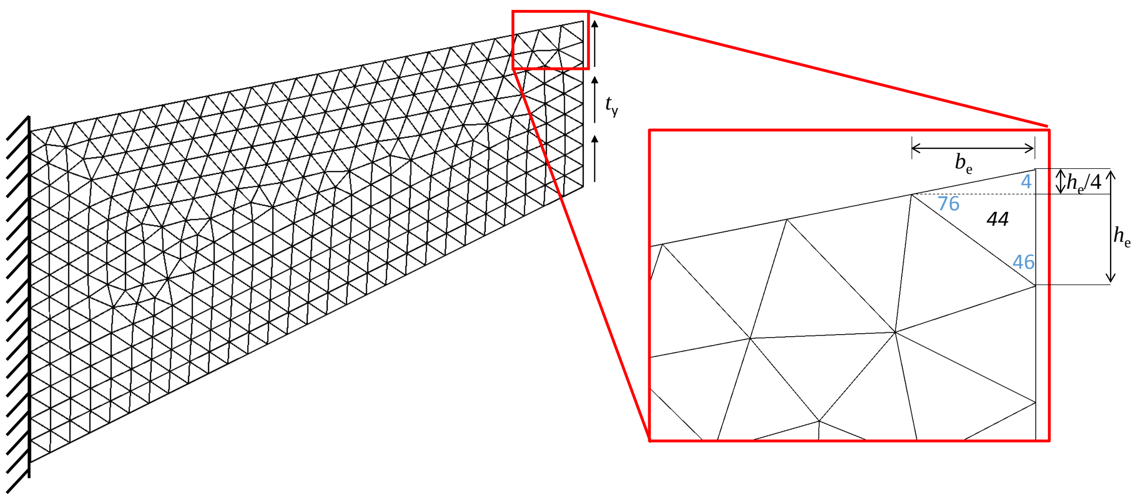


Figure 3: Illustration of the Cooks membrane problem (Problem 3)

The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\boldsymbol{v}\right)^{\mathrm{T}} \boldsymbol{D}\tilde{\nabla}\boldsymbol{u}\, t\, \mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\boldsymbol{b}\, t\, \mathrm{d}A + \int_{\mathcal{L}_g} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{t}\, t\, \mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{h}\, t\, \mathrm{d}\mathcal{L}$$

where A denotes the area of the specimen, t its thickness, $\mathcal{L}_g$ the part of the boundary with prescribed degrees of freedom ($\boldsymbol{g}$), $\mathcal{L}_h$ the part of the boundary with prescribed tractions ($\boldsymbol{h}$) and where $\boldsymbol{D}$ is the constitutive matrix relating stresses ($\boldsymbol{\sigma}$) and strains ($\boldsymbol{\varepsilon} = \tilde{\nabla}\boldsymbol{u}$) on Voigt form such that

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}.$$

The Cooks membrane problem is under the state of plain strain whereby the $\boldsymbol{D}-$matrix becomes

$$\boldsymbol{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}.$$

Tasks on the next page.

(a) Create your own sketch of the problem and indicate, specifically for the current problem, **the different types of boundaries and boundary conditions**. Then, **derive the FE-form of the problem using Galerkin's method.** Specify the contents (in general terms) of any matrices or vectors you introduce. No explicit expressions for shape functions or their derivatives are necessary in this part. **(1.0p)**

b) The specimen is meshed with linear triangular elements as indicated in the left part the figure. **Consider specifically element no 44** (with nodes 4, 76 & 46), indicated to the right, **and define a matrix expression for calculating its element stiffness matrix ($\boldsymbol{K}^e$)**. Here, you do not have to perform the actual matrix multiplications. But all matrix components in the resulting expression needs to be clearly defined for full point! You must also explain how any integrals are evaluated. **(1.0p)**

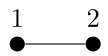
Here, it can be shown that the element stiffness matrix is only dependent on the relative difference of the initial nodal positions which means that for this subtask you can place the origin from which you define the nodal coordinates anywhere you want.

(c) Calculate the element load vector contribution from the applied traction on the right edge of element 44 and explain how this enters into the global load vector. You may have to suggest a numbering scheme for the degrees-of-freedom to solve this task fully. **(1.0p)**

1 Shape functions

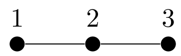
1.1 1D, linear

$$N_1^e = -\frac{1}{L}(x - x_2) \tag{1a}$$
$$N_2^e = \frac{1}{L}(x - x_1) \tag{1b}$$

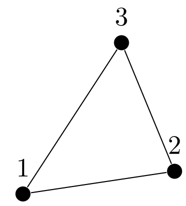


1.2 1D, quadratic

$$N_1^e = \frac{2}{L^2}(x - x_2)(x - x_3) \tag{2a}$$
$$N_2^e = -\frac{4}{L^2}(x - x_1)(x - x_3) \tag{2b}$$
$$N_3^e = \frac{2}{L^2}(x - x_1)(x - x_2) \tag{2c}$$



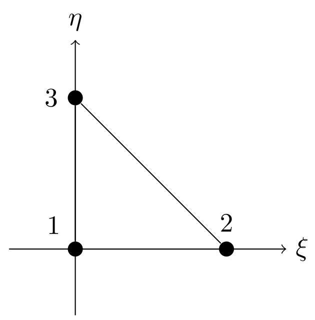
1.3 2D, linear triangle



$$N_1^e = \frac{1}{2A}(x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y) \tag{3a}$$
$$N_2^e = \frac{1}{2A}(x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y) \tag{3b}$$
$$N_3^e = \frac{1}{2A}(x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y) \tag{3c}$$

Parent element:

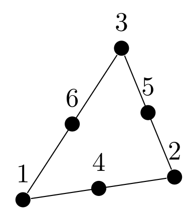
$$\overline{N}_1^e = 1 - \xi - \eta \tag{4a}$$
$$\overline{N}_2^e = \xi \tag{4b}$$
$$\overline{N}_3^e = \eta \tag{4c}$$

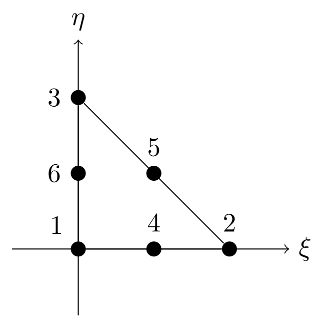


1.4 2D, Quadratic triangle

Parent element:

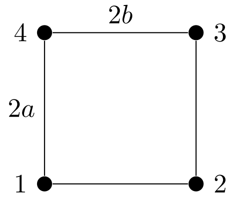
$$\overline{N}_1^e = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \tag{5a}$$
$$\overline{N}_2^e = \xi(2\xi - 1) \tag{5b}$$
$$\overline{N}_3^e = \eta(2\eta - 1) \tag{5c}$$
$$\overline{N}_4^e = 4\xi(1 - \xi - \eta) \tag{5d}$$
$$\overline{N}_5^e = 4\xi\eta \tag{5e}$$
$$\overline{N}_6^e = 4\eta(1 - \xi - \eta) \tag{5f}$$





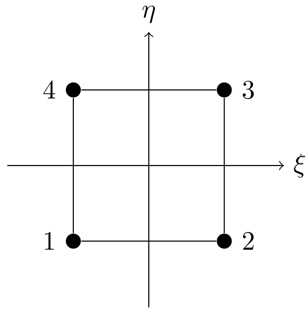
1.5 2D, bilinear

$$N_1^e = \frac{1}{4ab}(x - x_2)(y - y_4) \tag{6a}$$
$$N_2^e = -\frac{1}{4ab}(x - x_1)(y - y_3) \tag{6b}$$
$$N_3^e = \frac{1}{4ab}(x - x_4)(y - y_2) \tag{6c}$$
$$N_4^e = -\frac{1}{4ab}(x - x_3)(y - y_1) \tag{6d}$$



Parent element:

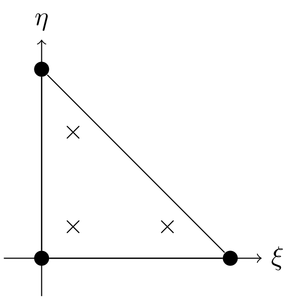
$$\overline{N}_1^e = \frac{1}{4}\,(\xi - 1)\,(\eta - 1) \tag{7a}$$
$$\overline{N}_2^e = -\frac{1}{4}\,(\xi + 1)\,(\eta - 1) \tag{7b}$$
$$\overline{N}_3^e = \frac{1}{4}\,(\xi + 1)\,(\eta + 1) \tag{7c}$$
$$\overline{N}_4^e = -\frac{1}{4}\,(\xi - 1)\,(\eta + 1) \tag{7d}$$



2 Gauss points

n	ξ_i	W_i
1	0.0000000000000000	2.0000000000000000
2	±0.5773502691896257	1.0000000000000000
3	0.0000000000000000	0.8888888888888889
	±0.7745966692414834	0.5555555555555556
4	±0.3399810435848563	0.6521451548625460
	±0.8611363115940525	0.3478548451374544

Table 1: Position of Gauss points ξ_i and corresponding weight W_i for n Gauss points.



n	(ξ_i, η_i)	W_i
1	$\left(\frac{1}{3}, \frac{1}{3}\right)$	$\frac{1}{2}$
3	$\left(\frac{1}{6}, \frac{1}{6}\right)$	$\frac{1}{6}$
	$\left(\frac{2}{3}, \frac{1}{3}\right)$	$\frac{1}{6}$
	$\left(\frac{1}{6}, \frac{2}{3}\right)$	$\frac{1}{6}$

3 Green-Gauss theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$.

$$\int_A \phi \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{w}\, \mathrm{d}A + \int_A (\boldsymbol{\nabla}\phi)^{\mathrm{T}} \boldsymbol{w}\, \mathrm{d}A = \int_{\mathcal{L}} \boldsymbol{n}^{\mathrm{T}} (\phi \boldsymbol{w})\, \mathrm{d}\mathcal{L} \tag{8}$$

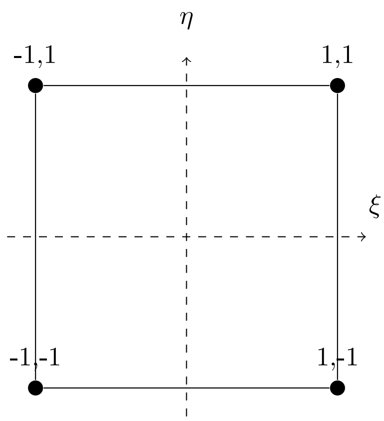
4 Gauss divergence theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$, $\text{div}(\boldsymbol{w}) = \nabla^{\mathrm{T}}\boldsymbol{w}$.

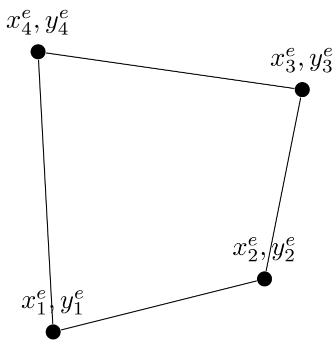
$$\int_A \boldsymbol{\nabla}^{\mathrm{T}}(\phi\boldsymbol{w})\ \mathrm{d}A = \int_{\mathcal{L}} (\phi\boldsymbol{w})^{\mathrm{T}}\,\boldsymbol{n}\,\mathrm{d}\mathcal{L}$$

5 Isoparametric mapping

Parent domain



Global domain



$$\mathbf{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}, \mathbf{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

$$x = x(\xi,\eta) = \overline{\mathbf{N}}^e(\xi,\eta)\,\mathbf{x}^e \tag{9}$$

$$y = y(\xi,\eta) = \overline{\mathbf{N}}^e(\xi,\eta)\,\mathbf{y}^e \tag{10}$$

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathbf{J}\begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial x} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial y} \end{bmatrix} = \left(\mathbf{J}^{\mathrm{T}}\right)^{-1}\begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \xi} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \eta} \end{bmatrix} \tag{12}$$

6 Matrix inversion

The inverse of the matrix $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is given by:

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})}\begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \quad \text{with } \det(\mathbf{M}) = M_{11}M_{22} - M_{12}M_{21}. \tag{13}$$

7 Stresses and strains

Hooke's generalised law: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

2D Strain-displ. relation: $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{bmatrix} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u},\ \boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix},\ \tilde{\boldsymbol{\nabla}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$

Problem 1a)

1a) Derive weak formulation of transient problem from $\tau = 0$ and onwards

[Strong Form - Not required by question!

→ $\rho c \dot{T} + \frac{d}{dx}(\cancel{A} q) = \cancel{Q}^{0}$ for $0 < x \leq L = h_1 + h_2 + h_3$, $\tau > 0$

→ $q(0) = -\alpha_{out}(T(0) - \bar{T})$

$q(L) = \alpha_{in}(T(L) - T_{in})$

or

$q_n(0) = \alpha_{out}(T(0) - \bar{T})$

$q_n(L) = \alpha_{in}(T(L) - T_{in})$

→ $T(x, \tau = 0) = T^0(x)$]

Weak Form

1. multiply by $v(x)$

$$v(x)\rho c \dot{T} + v(x)\frac{d}{dx}q = 0$$

2. Integrate over domain

$$\int_0^L v(x)\rho c \dot{T}\,dx + \underbrace{\int_0^L v(x)\frac{d}{dx}q\,dx}_{\text{integration by parts}} = 0$$

$$= [v(x)q(x)]_0^L - \int_0^L \frac{dv}{dx} q\,dx$$

$$\int_0^L v(x)\rho c \dot{T}\,dx - \int_0^L \frac{dv}{dx}\underbrace{q}_{-k\frac{dT}{dx}}\,dx = -[v(x)q(x)]_0^L$$

$$\int_0^L v(x)\rho c \dot{T}\,dx + \int_0^L \frac{dv}{dx}k\frac{dT}{dx} = -v(L)q(L) + v(0)q(0)$$

3. Insert natural BCs

$$\int_0^L v(x)\,\rho c \dot{T}\,dx + \int_0^L \frac{dv}{dx} k \frac{dT}{dx} = -v(L)\alpha_{in}(T(L) - T_{in}) - v(0)\,\alpha_{out}(T(0) - \bar{T})$$

<u>Weak Form</u>

$$\int_0^L v(x)\,\rho c \dot{T}\,dx + \int_0^L \frac{dv}{dx} k \frac{dT}{dx}\,dx + v(L)\alpha_{in} T(L) + v(0)\alpha_{out} T(0)$$

$$= v(L)\alpha_{in} T_{in} + v(0)\alpha_{out} \bar{T}$$

$$T(x, \tau = 0) = T^0$$

Grading instructions:
One significant mistake in derivations ==> -0.5p
Two or more significant mistake in derivations ==> -0.5p
Unclear statement of boundary conditions ==> -0.5p
Missing initial condition ==> -0.5p
Of course, points can not go below 0

1b) Derive, state semi-discrete FE-formulation

Introduce

- $T = \underline{N}\,\underline{a}$, $\quad \underline{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$, $\quad \underline{N} = [N_1 \; N_2 \; \dots \; N_n]$

  n: # of elements + 1

- $\dfrac{dT}{dx} = \dfrac{d}{dx}(\underline{N}\,\underline{a}) = \dfrac{d\underline{N}}{dx}\underline{a} = \underline{B}\,\underline{a}$, $\quad \underline{B} = [dN_1/dx \;\; dN_2/dx \; \dots]$

- $\dot{T} = \dfrac{dT}{d\tau} = \dfrac{d}{d\tau}(\underline{N}\,\underline{a}) = \underline{N}\,\dot{\underline{a}}$

- $v = \underline{N}\,\underline{c} = \underline{c}^T\underline{N}^T$

  $\underline{c}$: arbitrary

- $\dfrac{dv}{dx} = \underline{c}^T\underline{B}^T$

$$\underbrace{\int_0^L \underline{N}^T \rho c \underline{N}\, dx}_{\underline{C}}\, \dot{\underline{a}} + \underbrace{\int_0^L \underline{B}^T k \underline{B}\, dx}_{\underline{K}}\, \underline{a} + \dots$$

$$\dots + \underbrace{\left(\underline{N}^T(L)\,N(L)\,\alpha_{in} + \underline{N}^T(0)\,N(0)\,\alpha_{out}\right)}_{\underline{K}_c}\underline{a}$$

$$= \underbrace{\underline{N}^T(L)\,\alpha_{in} T_{in} + \underline{N}^T(0)\,\alpha_{out}\,\overline{T}}_{\underline{f}_b}$$

FE - Form

$$\begin{cases} \underline{C}\,\dot{\underline{a}} + (\underline{K} + \underline{K}_c)\,\underline{a} = \underline{f}_b \\ T(x, \tau = 0) = T^0 \end{cases}$$

Correction instructions:
- Missing specification of N and/or B ==> -0.25p
- Major mistakes ==> 0p
- don't take away additional points if again initial conditions are missing
- Clear definition of K, Kc, C and f is needed for full point

1c) How many elemets for stationary case.

→ consider the stationary differential equation

constant within each wall section

$$\frac{d}{dx}\left(Ak\frac{dT}{dx}\right) = 0$$

constand

→ integrate

$$Ak\frac{dT}{dx} = c_1$$

$$Ak\,T(x) = c_1 x + c_2$$

→ $T(x)$ will behave linearly, where the slope of the line is dependent on k

→ <u>3 elements</u> are required, one for each wall section.

Bad, wrong or no no motivation ==> 0p even if they answer 3 elements

1d) Does the number of elemets required for an accurate solution change for the transient case?

→ consider again the D.E.

$$\rho c\frac{dT}{d\tau} - \frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

where $T(x,\tau)$

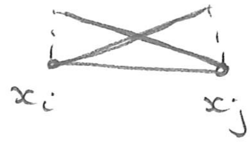
→ This is now a more complex partial differential equation. the solution of which is non-linear.

need to be clear for full point

→ for that purpose, <u>more element</u> will be required (A mesh convergence study in needed to say exactly how many). last Part not needed for full points

e) Based on your FE-formulation derive Task (b) calculate element capacity matrix for element with length $L_e = h_2/4$

for an element



$$N^e = \left[-\frac{1}{L_e}(x-x_j) \quad \frac{1}{L_e}(x-x_i) \right]$$

$$\underline{C}^e = \int_{x_i}^{x_j} \rho c \, \underline{N}^{eT} \underline{N}^e \, dx$$

$$= \rho c \int_{x_i}^{x_j} \begin{bmatrix} -\frac{1}{L_e}(x-x_j) \\ \frac{1}{L_e}(x-x_i) \end{bmatrix} \left[-\frac{1}{L_e}(x-x_j) \quad \frac{1}{L_e}(x-x_i) \right]$$

$$= \frac{\rho c}{L_e^2} \int_{x_i}^{x_j} \begin{bmatrix} (x-x_j)^2 & -(x-x_j)(x-x_i) \\ -(x-x_i)(x-x_j) & (x-x_i)^2 \end{bmatrix}$$

→ start upper left

$$\int_{x_i}^{x_j} (x-x_j)^2 \quad = \text{substitute} \quad s = x - x_j$$

$$\frac{ds}{dx} = 1 \quad \Rightarrow \quad ds = dx$$

$$s_i = x_i - x_j$$

$$s_j = x_j - x_j = 0$$

$$\Rightarrow \int_{x_i - x_j}^{0} s^2 \, ds = \left[\frac{1}{3} s^3 \right]_{x_i - x_j}^{0}$$

$$= -\frac{1}{3} (\underbrace{x_i - x_j}_{-L_e})^3$$

$$= L_e^3/3$$

→ upper right

$$-\int_{x_i}^{x_j} (x-x_j)(x-x_i)\, dx$$

$$= -\int_{x_i}^{x_j} x^2 - x_i x - x_j x + x_i x_j \, dx$$

$$= -\left[x^3/3 - x_i x^2/2 - x_j x^2/2 + x_i x_j x \right]_{x_i}^{x_j}$$

$$= -\Big[\cancel{x_j^3/3} - \cancel{x_i x_j^2/2} - \cancel{x_j^3/2} + \cancel{x_i x_j^2} - \big(\cancel{x_i^3/3} - \cancel{x_i^3/2}$$

$$- x_j x_i^2/2 + x_j x_i^2 \big)\Big]$$

$$= -\Big[x_j^3/3 - x_j^3/2 - x_i^3/3 + x_i^3/2 - x_i x_j^2/2 + x_i x_j^2$$

$$+ x_i^2 x_j/2 - x_i^2 x_j \Big]$$

$$= -\left[-x_j^3/6 + x_i^3/6 + x_i x_j^2/2 - x_i^2 x_j/2 \right]$$

$$= -\frac{1}{2}\left[-x_j^3/3 + x_i^3/3 + x_i x_j^2 - x_i^2 x_j \right]$$

$$= -\frac{1}{2}\left(\frac{1}{3} (\underbrace{x_i - x_j}_{-L_e})^3 \right)$$

$$= \frac{L_e^3}{6}$$

→ lower right

$$\int_{x_i}^{x_j} (x - x_i)^2 \quad \Rightarrow \text{sub:} \quad s = (x - x_i)$$

$$ds = dx$$

$$s_i = 0$$

$$s_j = x_j - x_i$$

$$\int_0^{x_j - x_i} s^2 \, ds = \left[s^3/3 \right]_0^{x_j - x_i}$$

$$= (x_j - x_i)^3 / 3$$

$$= L_e^3 / 3$$

so

$$C^e = \frac{\rho c}{\cancel{L_e^2}} \begin{bmatrix} L_e^{\cancel{3}}/3 & L_e^{\cancel{3}}/6 \\ L_e^{\cancel{3}}/6 & L_e^{\cancel{3}}/3 \end{bmatrix}$$

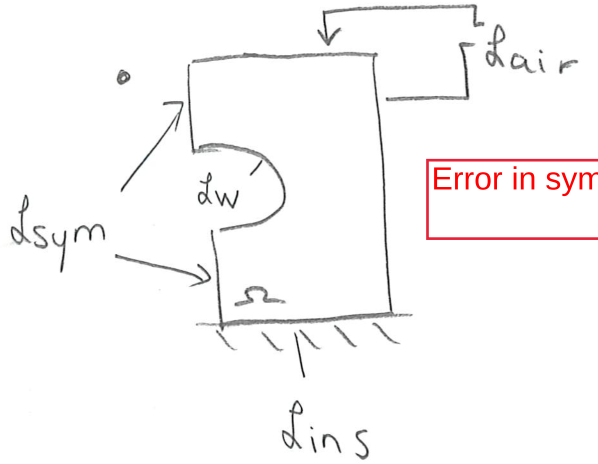
$$= \frac{\rho c L_e}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Minor error ok without point deduction but most should be correct for 0.5p. Otherwise 0p

Problem 2

2a). Determine smallest part of the cross section that can be analysed

· Derive the weak form



Error in symmetry (half domain must be considered) ==> -0.5p

Strong form

$$\nabla^T \underline{q} = 0 \quad \text{in } \Omega$$
$$q_n = 0 \quad \text{on } \mathcal{L}_{ins}$$
$$q_n = 0 \quad \text{on } \mathcal{L}_{sym}$$
$$q_n = \alpha(T - T_{air}) \quad \text{on } \mathcal{L}_{air}$$
$$q_n = \alpha(T - T_w) \quad \text{on } \mathcal{L}_w$$

Weak Form

1. Multiply by $v(x,y)$

$$v \nabla^T \underline{q} = 0$$

2. Integrate

$$\int_\Omega v \nabla^T \underline{q}\, d\Omega = 0$$

Green - Gauss → (8) in formula sheet

$$\int_\Omega v \nabla^T \underline{q}\, d\Omega = \oint_{\mathcal{L}} \underbrace{\underline{n}^T (v \underline{q})}_{= v \underbrace{\underline{q}^T \underline{n}}_{= q_n}}\, d\mathcal{L} - \int_\Omega (\nabla v)^T \underline{q}\, d\Omega$$

3. Insert natural BCs

$$\int_{\Omega} (\nabla v)^T \underline{q} \, d\Omega = \oint_{\mathcal{L}} v \, q_n \, d\mathcal{L}$$

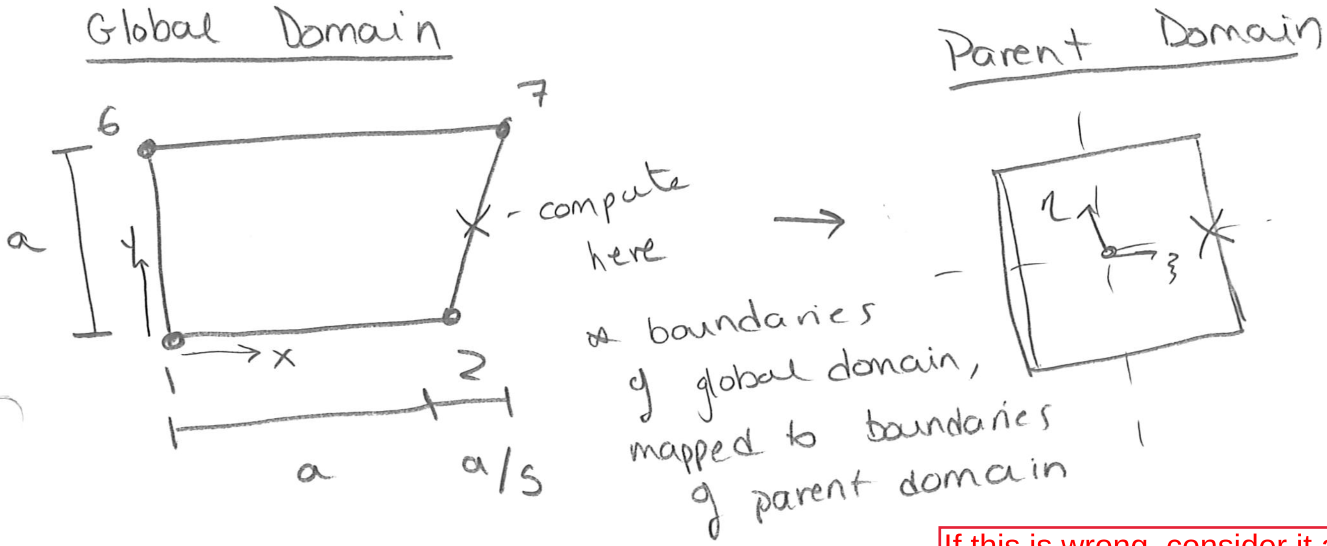
$$= \int_{\mathcal{L}_{ins}} v \cancel{q_n}^{\,0} \, d\mathcal{L} + \int_{\mathcal{L}_{air}} v \, q_n \, d\mathcal{L} + \int_{\mathcal{L}_{sym}} v \cancel{q_n}^{\,0} \, d\mathcal{L} + \int_{\mathcal{L}_{W}} v \, q_n \, d\mathcal{L}$$

$$= \int_{\mathcal{L}_{air}} v \alpha (T - T_{air}) \, d\mathcal{L} + \int_{\mathcal{L}_{W}} v \alpha (T - T_W) \, d\mathcal{L}$$

(could also include Fourier's law
→ not needed for full points)

One significant mistake in derivations ==> -0.5p
Two or more significant mistakes in derivations ==> 0p
Missing clarity in how natural (Neumann) BCs enter the weak form ==> -0.5p

2b) compute Jacobian for lower left element @ midpoint on right edge



° compute J at $(\xi, \eta) = (1, 0)$

If this is wrong, consider it as -0.25p (but in the end we only give points rounded to the next half point (2.25 => 2.5p)

$$J = \begin{bmatrix} dx/d\xi & dx/d\eta \\ dy/d\xi & dy/d\eta \end{bmatrix}$$

$$\frac{dx}{d\xi} = \frac{d\underline{N}^e(\xi,\eta)}{d\xi}\,\underline{x}^e \qquad \frac{dy}{d\xi} = \frac{d\underline{N}^e}{d\xi}\,\underline{y}^e$$

$$\frac{dx}{d\eta} = \frac{d\underline{N}^e(\xi,\eta)}{d\eta}\,\underline{x}^e \qquad \frac{dy}{d\eta} = \frac{d\underline{N}^e}{d\eta}\,\underline{y}^e$$

$$\rightarrow \underline{x}^e = \begin{bmatrix} 0 \\ a \\ 6a/5 \\ 0 \end{bmatrix} \qquad \underline{y}^e = \begin{bmatrix} 0 \\ 0 \\ a \\ a \end{bmatrix}$$

Be careful in the way nodes are ordered and xe and ye are defined. Obvious lack of understanding ==> -0.5p

* be carful with the order!

$$\frac{dN}{d\xi} = \begin{bmatrix} \frac{1}{4}(\eta-1) & -\frac{1}{4}(\eta-1) & \frac{1}{4}(\eta+1) & -\frac{1}{4}(\eta+1) \end{bmatrix}$$

$$\Rightarrow \left.\frac{dN}{d\xi}\right|_{(1,0)} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\frac{dN}{d\eta} = \begin{bmatrix} \frac{1}{4}(\xi-1) & -\frac{1}{4}(\xi+1) & \frac{1}{4}(\xi+1) & -\frac{1}{4}(\xi-1) \end{bmatrix}$$

$$\Rightarrow \left.\frac{dN}{d\eta}\right|_{(1,0)} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$\frac{dx}{d\xi} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ a \\ 6a/5 \\ 0 \end{bmatrix}$$

$$= 11a/20$$

$$\frac{dx}{d\eta} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ a \\ 6a/5 \\ 0 \end{bmatrix}$$

$$= \frac{a}{10}$$

$$\frac{dy}{d\xi} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a \\ a \end{bmatrix}$$

$$= 0$$

$$\frac{dy}{d\eta} = \begin{bmatrix} 0 & -1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ a \\ a \end{bmatrix}$$

$$= \frac{a}{2}$$

$$J = \begin{bmatrix} 11a/20 & a/10 \\ 0 & a/2 \end{bmatrix}$$

mistake in determinant calculation (if everything is correct) can still give full point (unless the error is major)

$$\det(J) = 11a/20 \cdot a/2 - \cancel{\frac{a}{10}} \cdot 0 \xrightarrow{} 0$$

$$= \frac{11a^2}{40}$$

2c) Given the solution

$T_1 = 3°C$
$T_2 = 4°C$
$T_6 = 2°C$
$T_7 = 5°C$

set up the expression to compute the temperature gradien ∇T @ same point as in 2b)

$$\underline{\nabla T} = \underline{\nabla}(\underline{N}^e \underline{a}^e)$$

$$= \underline{B}^e \underline{a}^e \rightarrow \begin{bmatrix} 3° \\ 4° \\ 5° \\ 2° \end{bmatrix}$$

Somewhere, the derivatives need to be defined

$$\begin{bmatrix} dN_1/dx & dN_2/dx & dN_3/dx & dN_4/dx \\ dN_1/dy & dN_2/dy & dN_3/dy & dN_4/dy \end{bmatrix} = \begin{bmatrix} d\underline{N}^e/dx \\ d\underline{N}^e/dy \end{bmatrix}$$

if Be for elasticity problem is used, automatically at least -0.5p

$$\begin{bmatrix} d\underline{N}^e/dx \\ d\underline{N}^e/dy \end{bmatrix} = (J^T)^{-1} \begin{bmatrix} d\underline{N}^e/d\xi \\ d\underline{N}^e/d\eta \end{bmatrix}$$

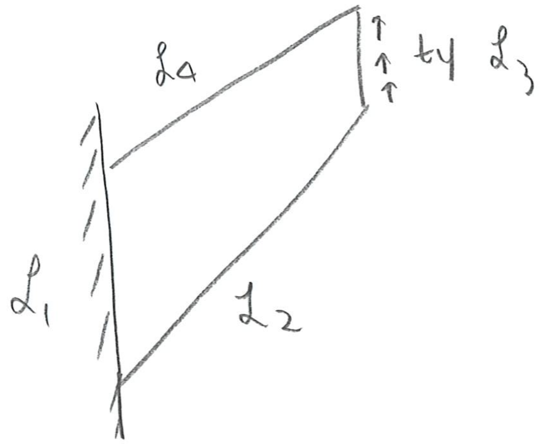
Computed in 2b)

Computed in 2b)

No need to actually invert the Jacobian or multiply matrices together. As a result, errors in this should not be considered

Problem 3

3a) Create your own sketch of the problem, indicate boundary conditions. Derive the FE-form.



BCs

$$\underline{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ on } \mathcal{L}_1$$

$$\underline{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ on } \mathcal{L}_2, \mathcal{L}_4$$

$$\underline{t} = \begin{bmatrix} 0 \\ t_y \end{bmatrix} \text{ on } \mathcal{L}_3$$

FE - Form

let $u = \underline{N}\,\underline{a}$, $\underline{a} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{2n} \end{bmatrix}$

$$\underline{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n \\ 0 & N_1 & 0 & N_2 & \cdots & 0 \end{bmatrix}$$

N_n: # of nodes

Needs to be defined

$$\tilde{\nabla} = \begin{bmatrix} d/dx & 0 \\ 0 & d/dy \\ d/dy & d/dx \end{bmatrix}$$

$$\underline{\tilde{\nabla}} u = \underline{B}\,\underline{a}$$

$$\underline{B} = \begin{bmatrix} dN_1/dx & 0 & dN_2/dx & 0 & \cdots \\ 0 & dN_1/dx & 0 & dN_2/dx & \cdots \\ dN_1/dy & dN_1/dx & dN_2/dy & dN_2/dx & \cdots \end{bmatrix}$$

Needs to be defined

$$v = \underline{N}\,\underline{c}$$

$$\underline{\tilde{\nabla}} v = \underline{B}\,\underline{c}$$

$$\underbrace{\int_A \underline{B}^T \underline{D}\, \underline{B}\, t\, dA}_{\underline{K}}\, \underline{a} = \underbrace{\int_A \underline{N}^T \underline{b}\, t\, dA}_{\underline{f}_e} + \underbrace{\int_{\mathcal{L}} \underline{N}^T \underline{t}\, t\, d\mathcal{L}}$$

$$\underline{f}_b \left\{ \begin{aligned} &= \int_{\mathcal{L}_1} \underline{N}^T \underline{t}\, t\, d\mathcal{L} + \cancel{\int_{\mathcal{L}_2} \underline{N}^T \underbrace{\underline{t}}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} t\, d\mathcal{L}}^{\,0} \\ &+ \int_{\mathcal{L}_3} \underline{N}^T \underbrace{\underline{t}}_{\begin{bmatrix} 0 \\ t_y \end{bmatrix}} t\, d\mathcal{L} + \cancel{\int_{\mathcal{L}_4} \underline{N}^T \underbrace{\underline{t}}_{\begin{bmatrix} 0 \\ 0 \end{bmatrix}} t\, d\mathcal{L}}^{\,0} \end{aligned} \right.$$

$$\begin{cases} \underline{K}\underline{a} = \underline{f}_l + \underline{f}_b \\ \underline{u} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ on } \mathcal{L}_1 \end{cases}$$

One significant mistake in derivation ==> -0.5p
Two significant mistakes in derivation ==> 0p

Needs to be defined, if not ==> -0.5p

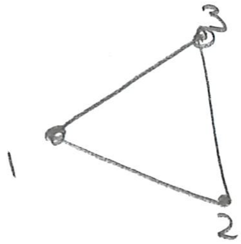
3b) Define an expression for $\underline{K}^e$ for element no 44.

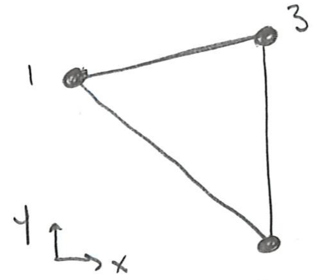
$$\underline{K}^e = \int_{A^e} \underline{B}^{eT} \underline{D}\, \underline{B}^e \, dA$$

$\downarrow$

determine B^e

For 2D linear tringle





$(x_1, y_1) = (0, h_e - h_e/4)$

$(x_2, y_2) = (b_e, 0)$

$(x_3, y_3) = (b_e, h_e)$

$$N_1^e = \frac{1}{2A}\left(x_2 y_3 - x_3 y_2 + (y_2 - y_3)x + (x_3 - x_2)y\right)$$

$$\frac{dN_1^e}{dx} = \frac{1}{2A}(y_2 - y_3) = -\frac{h_e}{2A}$$

$$\frac{dN_1^e}{dy} = \frac{1}{2A}(x_3 - x_2) = 0$$

$$\frac{dN_2^e}{dx} = \frac{1}{2A}(y_3 - y_1) = \frac{h_e}{8A}$$

$$\frac{dN_2^e}{dy} = \frac{1}{2A}(x_1 - x_3) = -\frac{b_e}{2A}$$

$$\frac{dN_3^e}{dx} = \frac{1}{2A}(y_1 - y_2) = \frac{h_e - h_e/4}{2A}$$

$$\frac{dN_3^e}{dy} = \frac{1}{2A}(x_2 - x_1) = \frac{b_e}{2A}$$

$$\underline{B}^e = \frac{1}{2A}\begin{bmatrix} -h_e & 0 & h_e/4 & 0 & h_e - h_e/4 & 0 \\ 0 & 0 & 0 & -b_e & 0 & b_e \\ 0 & -he & -be & he/4 & be & he-he/4 \end{bmatrix}$$

0.5p

$$\underline{D} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix}$$

$$\underline{K}^e = \int_{A^e} \underbrace{\underline{B}^{eT}\,\underline{D}\,\underline{B}^e}\,dA$$

0.25p

$\downarrow$

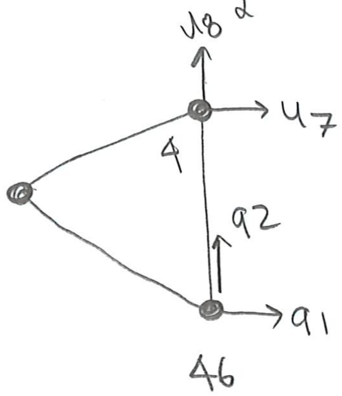
all constant $\Rightarrow$ no dependence on x or y

$$\underline{K}^e = \underline{B}^{eT}\,\underline{D}\,\underline{B}^e \underbrace{\int_{A^e} 1\,dA}_{= A^e} = \frac{h_e \cdot b_e}{2}$$

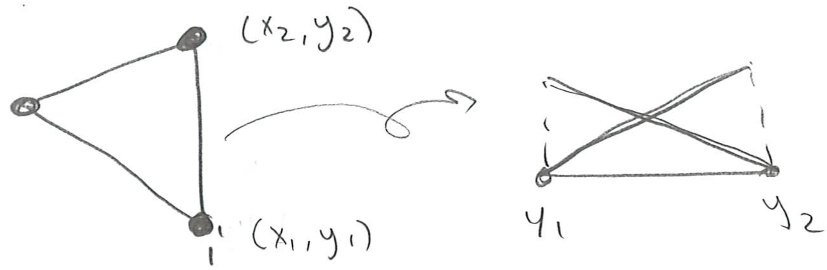
0.25p

3c) Compute load vector contribution from applied traction of element ~~44~~.
Explain how it is assembled.

$$\underline{f}_b^e = \int_{\ell^e} \underline{N}^{eT} \begin{bmatrix} 0 \\ t_y \end{bmatrix} t \, d\ell$$



Global



local

$$\underline{N}^{eT} = \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \end{bmatrix}$$

$0 = N_3^e$ on boundary

$N_3^e = 0$ on boundary

$$N_1 = -\frac{1}{L^e}(y - y_2)$$

$$N_2 = \frac{1}{L^e}(y - y_1)$$

$$\underline{f}_b^e = \int_{y_1}^{y_2} \begin{bmatrix} N_1^e & 0 \\ 0 & N_1^e \\ N_2^e & 0 \\ 0 & N_2^e \end{bmatrix} \begin{bmatrix} 0 \\ t_y \end{bmatrix} t \, dy$$

$$= \int_{y_1}^{y_2} \begin{bmatrix} 0 \\ N_1^e t_y \\ 0 \\ N_2^e t_y \\ 0 \\ 0 \end{bmatrix} t \, dy$$

$$= \frac{t_y \cdot t}{L_e} \int_{y_1}^{y_2} \begin{bmatrix} -(y - y_2) \\ (y - y_1) \end{bmatrix} dy$$

(only consider non zero components for now)

$$= \frac{t_y \cdot t}{L_e} \begin{bmatrix} -(y^2/2 - y_2 y) \\ (y^2/2 - y_1 y) \end{bmatrix}_{y_1}^{y_2}$$

$$= \frac{t_y \cdot t}{L_e} \begin{bmatrix} -(y_2^2/2 - y_2^2 - y_1^2/2 + y_1 y_2) \\ (y_2^2/2 - y_1 y_2 - y_1^2/2 + y_1^2) \end{bmatrix}$$

$$= \frac{t_y \cdot t}{L_e} \begin{bmatrix} -(-y_2^2/2 - y_1^2/2 + y_1 y_2) \\ (y_2^2/2 + y_1^2/2 - y_1 y_2) \end{bmatrix}$$

0.5p for the correct integration

$$= \frac{t_y \cdot t}{L_e} \begin{bmatrix} y_2^2/2 + y_1^2/2 - y_1 y_2 \\ y_2^2/2 + y_1^2/2 - y_1 y_2 \end{bmatrix} \Big\} \rightarrow \frac{1}{2} \big(\underbrace{y_2 - y_1}_{L_e} \big)^2$$

$$\Rightarrow \frac{t_y \cdot t \cdot L_e}{2} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{array}{l} \\ \rightarrow \text{assemble into } 92^{nd} \text{ D.O.F} \\ \\ \rightarrow \text{assemble into } 8^{th} \text{ D.O.F.} \\ \\ \\ \end{array}$$

0.5p for the correct definition of the 6x1 vector