VSM167 Finite element method - basics
Re-exam 2018-04-06, 14:00-18:00

Instructor:	Magnus Ekh (phone 070-828 2358). The instructor will visit the exam around 15:00 and 16:30.
Solution:	Example solutions will be posted within a few days after the exam on the course homepage.
Grading:	The grades will be reported to the registration office on Wednesday 25 April 2018 the latest.
Review:	It will be possible to review the grading at the Division of Material and Computational Mechanics (floor 3 in M-building) on Friday 27 April 12:00-13:00 and Friday 4 May 12:00-13:00.
Permissible aids:	No aids. **Note**: A formula sheet is appended to this exam thesis.

Problem 1

Consider the 1 dimensional bar supported on its left end with a spring with stiffness k [N/m], subjected to a force F on its right end, see Figure 1 (top) below.

The weak form of the governing equation to this problem reads:

$$\int_0^L \frac{dv}{dx} AE \frac{du}{dx} dx = \left[vAE\frac{du}{dx} \right]_0^L$$

where the strain is given by $\varepsilon = \frac{du}{dx}$ and the stress by $\sigma = E\varepsilon$.

Furthermore, the mesh for the FE stress analysis is also shown in Figure 1 (bottom). The three elements and four nodes (the nodes have the same numbering as the degrees of freedom) are numbered as shown in the figure.

The resulting discretised system of FE equations will be

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$$

where $\boldsymbol{K}$ is the global stiffness matrix for the problem, $\boldsymbol{a}$ is a vector containing the displacement degrees of freedom and $\boldsymbol{f}$ is the load vector.

The element stiffness matrices for the three elements (linear basis functions) are given as follows:

$$\boldsymbol{K}_1^e = \begin{bmatrix} a & -a \\ -a & a \end{bmatrix}, \quad \boldsymbol{K}_2^e = \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}, \quad \boldsymbol{K}_3^e = \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}$$

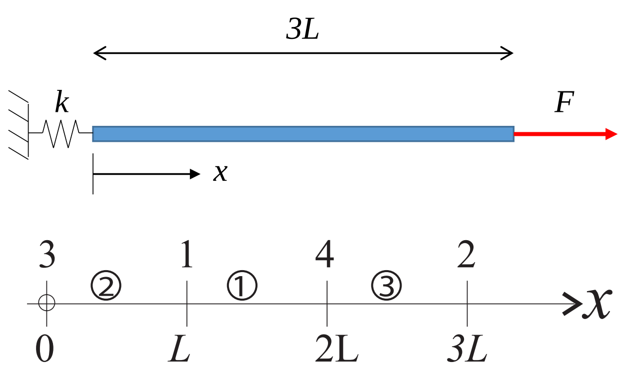


Figure 1: The one dimensional bar considered in Problem 1 (top) and the corresponding 1D mesh with three elements and four nodes (bottom). (Problem 1)

(a) Assemble the global stiffness matrix $\boldsymbol{K}$ without influence from the boundary conditions. **(1p)**

Continued on the next page!

(b) The global FE approximation $u(x) = \boldsymbol{N}(x)\boldsymbol{a}$ is assumed where $\boldsymbol{N}$ contains the global shape functions. If the solution of a particular problem was computed as

$$\boldsymbol{a} = \begin{bmatrix} 0.7 \\ 1.5 \\ 0.2 \\ 1.2 \end{bmatrix},$$

then determine $\boldsymbol{N}(x)$ and $u(x)$ at $x = 4L/3$. **(1p)**

(c) Explain how the spring support enters into the discretised FE equations and how it affects $\boldsymbol{K}$ and $\boldsymbol{f}$. **(1p)**

Not for MHA021

Problem 2

A wooden board is lying on a wet surface as shown in Figure 2. The free sides of the board is subjected to convective heat transfer with a constant ambient temperature T_{air}, whereas the bottom side of the board has the same temperature as the wet surface T_{ground} (constant in time). The board has absorbed moisture from the bottom, which affects the heat capacity c_p and the thermal conductivity k, making them function of space according to the specification below.

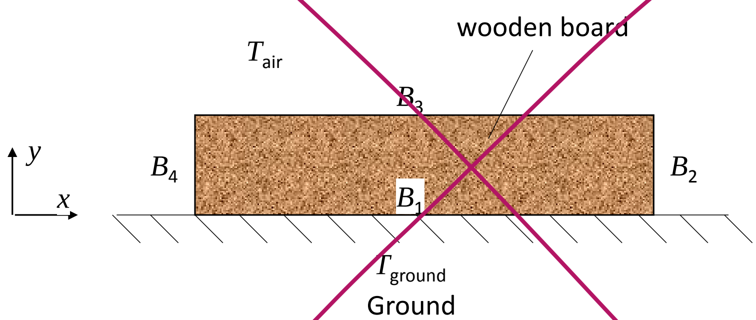


Figure 2: Transient heat flow problem. (Problem 2)

We here consider the transient heat flow for a cross section of unit thickness (due to the length of the board, the flux out of plane can be neglected), whereby the temperature $T(x, y, \tau)$ is given as

$$\begin{aligned} -\text{div}(\boldsymbol{q}) &= \rho c_p(x,y)\frac{\partial T}{\partial \tau} \quad \text{in } A \quad \text{for } 0 < \tau < \tau_{\text{end}}, \\ T(x,y,\tau) &= T_{\text{ground}} \quad \text{on } B_1, \\ q_n &= \alpha(T(x,y,\tau) - T_{\text{air}}) \quad \text{on } B_2, B_3, B_4, \\ T(x,y,0) &= T_{\text{ground}}. \end{aligned}$$

The specific heat capacity and thermal conductivity are functions of space as

$$c_p(x,y) = c_0\left(1 + e^{-c_1 y}\right), \qquad k(x,y) = k_0\left(1 + e^{-k_1 y}\right). \tag{1}$$

Furthermore, ρ, α, k_0, k_1, c_0, c_1 are known positive constants. The wood material is assumed to be isotropic (w.r.t heat flow) and obey Fourier's law, $\boldsymbol{q} = -k(x,y)\boldsymbol{\nabla}T$.

(a) Utilise that the problem is symmetric and derive the weak form for the smallest possible subdomain of the problem at hand. You may assume a unit thickness to simplify the derivations. **(1.0p)**

(b) Starting from the weak form, derive the semi-discrete form of the problem by introducing the approximation in space on the form $T(x, y, \tau) \approx \boldsymbol{N}(x, y)\boldsymbol{a}(\tau)$, i.e., derive the expressions for the matrices $\boldsymbol{C}$ and $\boldsymbol{K}$ and the vector $\boldsymbol{f}$ in the expression.

$$\boldsymbol{C}\dot{\boldsymbol{a}} + (\boldsymbol{K} + \boldsymbol{K}_c)\boldsymbol{a} = \boldsymbol{f}, \quad \text{for } 0 < t < t_{\text{end}}. \qquad \textbf{(1p)}$$

(Continued on next page)

(c) An engineer fails to identify the possibility to simplify the problem with respect to symmetry and chooses to solve for the whole domain. Some results from his simulation ($\Delta\tau = 50$ s, $\Theta = 0.5$, $\tau_{\text{end}} = 50$ min) are shown in Figure 3 for $T_{\text{ground}} = 0$ °C, $T_{\text{air}} = 20$ °C, with constants as: $\rho = 450$ kg/m^3, $\alpha = 7.67$ W/(m^2 °C), $k_0 = 0.1$ W/(m °C), $k_1 = 10$ 1/m, $c_0 = 1,400$ J/(kg °C), $c_1 = 10$ 1/m. Are the results reasonable? Motivate your answer! **(1p)**

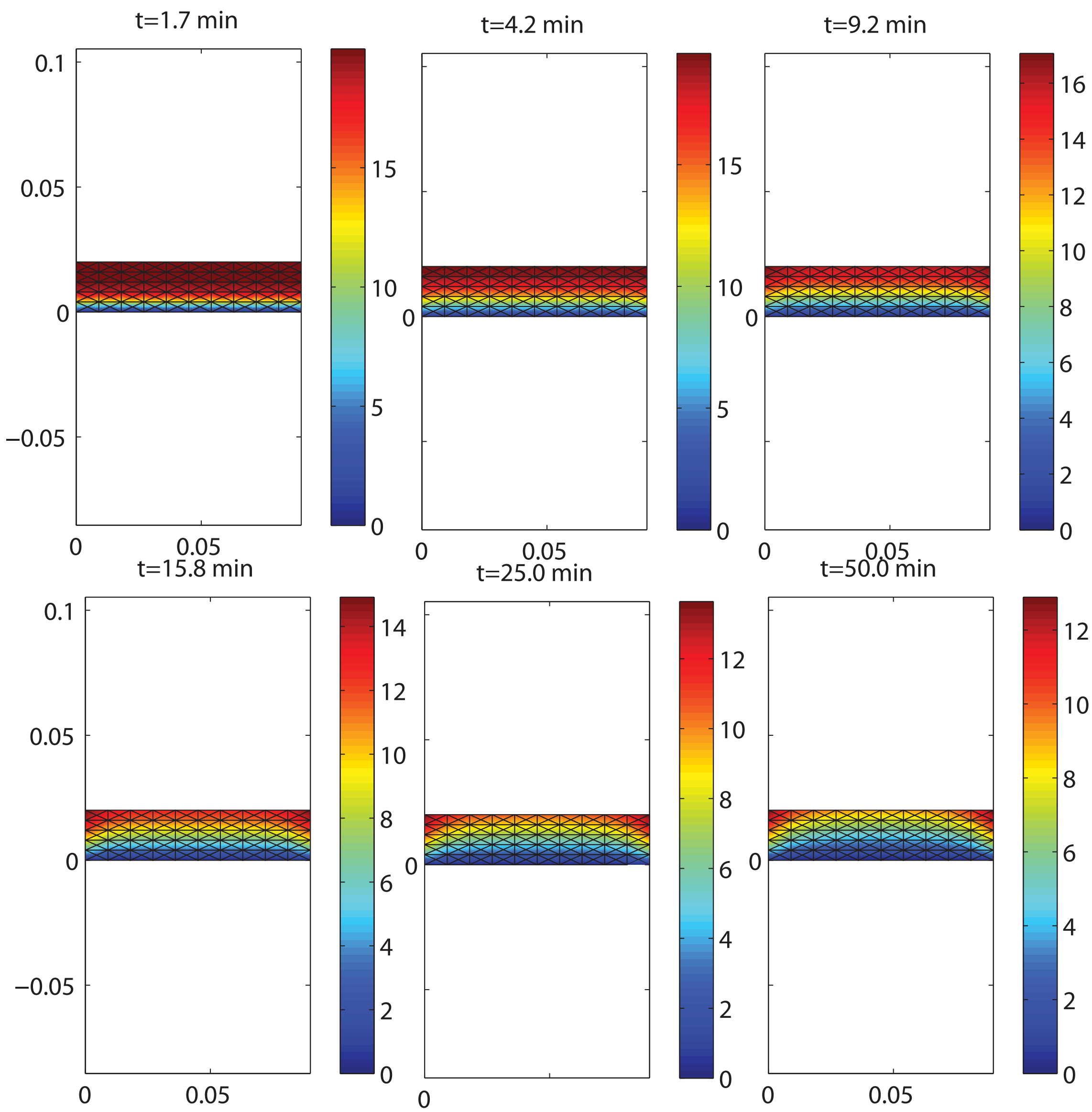
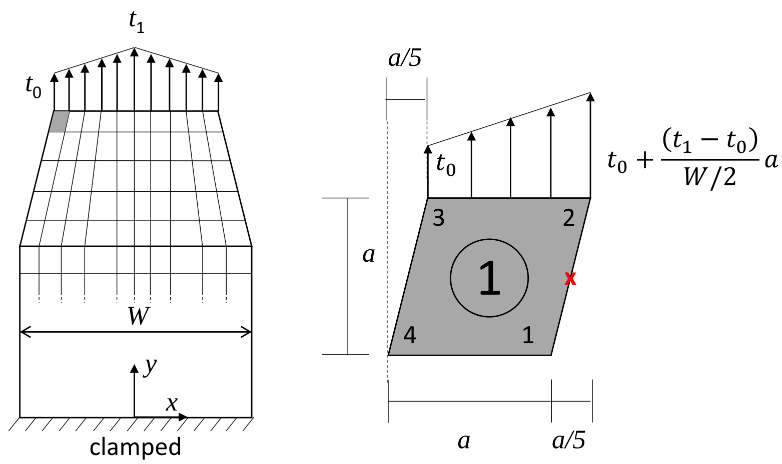


Figure 3: Transient heat flow solution. Temperature scales in °C. Please note that $t = \tau$ is the time in the pictures. (Problem 2)

Problem 3

Consider a thin metal specimen as sketched in Figure 4a loaded in tension by a linearly varying traction (left-to-right: from t_0 to t_1 and then back to t_0) acting in the $y-$direction along the upper boundary at the same time as the lower boundary is clamped (prescribed displacements $u_x = u_y = 0$).



a) *b)*

Figure 4: a) A thin specimen loaded in tension with prescribed traction along the top edge and a clamped lower boundary. b) A close up of element no 1 (marked in grey in part *a*) with dimensions, node numbering and applied traction indicated. The point where the Jacobian matrix associated with the isoparametric mapping is to be evaluated is to marked with a red cross. (Problem 3)

The governing 2D elasticity equation on weak form for this problem is generally given by:

$$\int_A \left(\tilde{\nabla}\boldsymbol{v}\right)^{\mathrm{T}} \boldsymbol{D}\tilde{\nabla}\boldsymbol{u}\, t \,\mathrm{d}A = \int_A \mathbf{v}^{\mathrm{T}}\boldsymbol{b}\, t\,\mathrm{d}A + \int_{\mathcal{L}_g} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{t}\, t\,\mathrm{d}\mathcal{L} + \int_{\mathcal{L}_h} \boldsymbol{v}^{\mathrm{T}}\boldsymbol{h}\, t\,\mathrm{d}\mathcal{L}$$

where A denotes the area of the specimen, t its thickness, $\mathcal{L}_g$ the part of the boundary with prescribed degrees of freedom ($\boldsymbol{g}$), $\mathcal{L}_h$ the part of the boundary with prescribed tractions ($\boldsymbol{h}$) and where $\boldsymbol{D}$ is the constitutive matrix relating stresses ($\boldsymbol{\sigma}$) and strains ($\boldsymbol{\varepsilon} = \tilde{\nabla}\boldsymbol{u}$) on Voigt form such that

$$\boldsymbol{\sigma} = \boldsymbol{D}\boldsymbol{\varepsilon}.$$

Tasks on the next page.

(a) Create your own sketch of the problem and indicate, specifically for the current problem, **the different types of boundaries and boundary conditions**. Then, **derive the FE-form of the problem using Galerkin's method.** Specify the contents (in general terms) of any matrices or vectors you introduce. No explicit expressions for shape functions or their derivatives are necessary in this part. **(1.0p)**

b) The specimen is meshed with isoparametric quadrilateral elements as indicated in part *a*) of the figure. Consider specifically element no 1, indicated in grey and also enlarged in part *b*) of the figure. **For the isoparametric mapping of element no 1, calculate the Jacobian matrix $\boldsymbol{J}$ in the midpoint of the right edge between nodes 1 and 2** (corresponding to isoparametric coordinates $\xi = 1$ and $\eta = 0$ and marked with a red cross in Figure 4b). **(1.0p)**

Here, it can be shown that the Jacobian matrix is dependent only on the relative difference of the initial nodal positions which means that for this subtask you can place the origin from which you define the nodal coordinates anywhere you want.

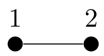
(c) Calculate the element load vector contribution from the applied traction on element 1 and explain how this enters into the global load vector. You may have to suggest a numbering scheme for the dgrees-of-freedom to solve this task fully. **(1.0p)**

1 Shape functions

1.1 1D, linear

$$N_1^e = -\frac{1}{L}(x - x_2) \tag{1a}$$

$$N_2^e = \frac{1}{L}(x - x_1) \tag{1b}$$

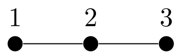


1.2 1D, quadratic

$$N_1^e = \frac{2}{L^2}(x - x_2)(x - x_3) \tag{2a}$$

$$N_2^e = -\frac{4}{L^2}(x - x_1)(x - x_3) \tag{2b}$$

$$N_3^e = \frac{2}{L^2}(x - x_1)(x - x_2) \tag{2c}$$

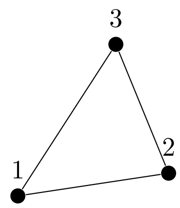


1.3 2D, linear triangle

$$N_1^e = \frac{1}{2A}(x_2y_3 - x_3y_2 + (y_2 - y_3)x + (x_3 - x_2)y) \tag{3a}$$

$$N_2^e = \frac{1}{2A}(x_3y_1 - x_1y_3 + (y_3 - y_1)x + (x_1 - x_3)y) \tag{3b}$$

$$N_3^e = \frac{1}{2A}(x_1y_2 - x_2y_1 + (y_1 - y_2)x + (x_2 - x_1)y) \tag{3c}$$

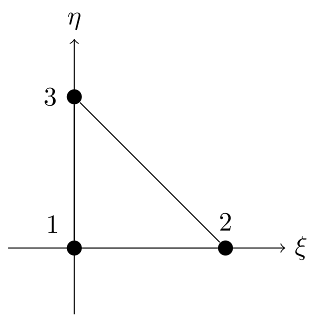


Parent element:

$$\overline{N}_1^e = 1 - \xi - \eta \tag{4a}$$

$$\overline{N}_2^e = \xi \tag{4b}$$

$$\overline{N}_3^e = \eta \tag{4c}$$



1.4 2D, Quadratic triangle

Parent element:

$$\overline{N}_1^e = (1 - \xi - \eta)(1 - 2\xi - 2\eta) \tag{5a}$$

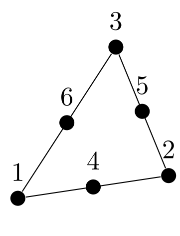
$$\overline{N}_2^e = \xi(2\xi - 1) \tag{5b}$$

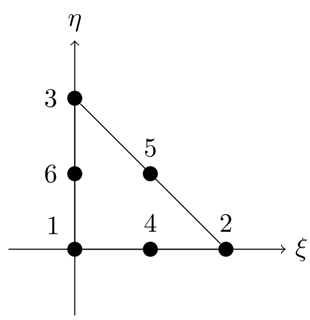
$$\overline{N}_3^e = \eta(2\eta - 1) \tag{5c}$$

$$\overline{N}_4^e = 4\xi(1 - \xi - \eta) \tag{5d}$$

$$\overline{N}_5^e = 4\xi\eta \tag{5e}$$

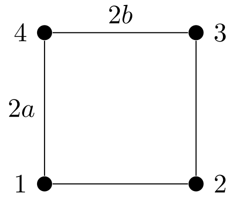
$$\overline{N}_6^e = 4\eta(1 - \xi - \eta) \tag{5f}$$





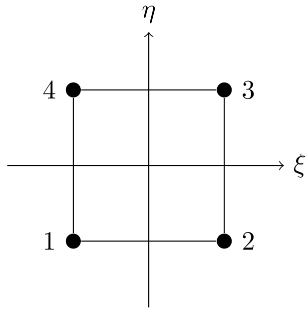
1.5 2D, bilinear

$$N_1^e = \frac{1}{4ab}(x - x_2)(y - y_4) \tag{6a}$$
$$N_2^e = -\frac{1}{4ab}(x - x_1)(y - y_3) \tag{6b}$$
$$N_3^e = \frac{1}{4ab}(x - x_4)(y - y_2) \tag{6c}$$
$$N_4^e = -\frac{1}{4ab}(x - x_3)(y - y_1) \tag{6d}$$



Parent element:

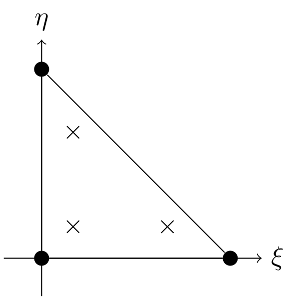
$$\overline{N}_1^e = \frac{1}{4}\left(\xi - 1\right)\left(\eta - 1\right) \tag{7a}$$
$$\overline{N}_2^e = -\frac{1}{4}\left(\xi + 1\right)\left(\eta - 1\right) \tag{7b}$$
$$\overline{N}_3^e = \frac{1}{4}\left(\xi + 1\right)\left(\eta + 1\right) \tag{7c}$$
$$\overline{N}_4^e = -\frac{1}{4}\left(\xi - 1\right)\left(\eta + 1\right) \tag{7d}$$



2 Gauss points

n	ξ_i	W_i
1	0.0000000000000000	2.0000000000000000
2	±0.5773502691896257	1.0000000000000000
3	0.0000000000000000	0.8888888888888889
	±0.7745966692414834	0.5555555555555556
4	±0.3399810435848563	0.6521451548625460
	±0.8611363115940525	0.3478548451374544

Table 1: Position of Gauss points ξ_i and corresponding weight W_i for n Gauss points.



n	(ξ_i, η_i)	W_i
1	$\left(\frac{1}{3}, \frac{1}{3}\right)$	$\frac{1}{2}$
3	$\left(\frac{1}{6}, \frac{1}{6}\right)$	$\frac{1}{6}$
	$\left(\frac{2}{3}, \frac{1}{3}\right)$	$\frac{1}{6}$
	$\left(\frac{1}{6}, \frac{2}{3}\right)$	$\frac{1}{6}$

3 Green-Gauss theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$.

$$\int_A \phi \boldsymbol{\nabla}^{\mathrm{T}} \boldsymbol{w} \, \mathrm{d}A + \int_A (\boldsymbol{\nabla}\phi)^{\mathrm{T}} \boldsymbol{w} \, \mathrm{d}A = \int_{\mathcal{L}} \boldsymbol{n}^{\mathrm{T}} (\phi \boldsymbol{w}) \, \mathrm{d}\mathcal{L} \tag{8}$$

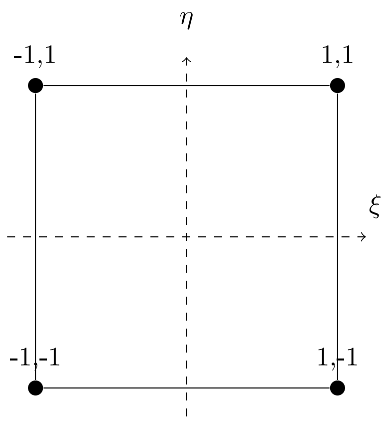
4 Gauss divergence theorem

$\boldsymbol{w}$ = vector field, ϕ = scalar field, $\boldsymbol{n}$ = normal to $\mathcal{L}$, $\text{div}(\boldsymbol{w}) = \nabla^{\mathrm{T}}\boldsymbol{w}$.

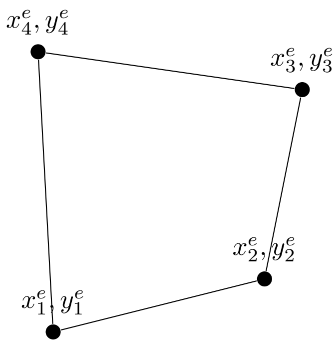
$$\int_A \boldsymbol{\nabla}^{\mathrm{T}}(\phi \boldsymbol{w})\, \mathrm{d}A = \int_{\mathcal{L}} (\phi \boldsymbol{w})^{\mathrm{T}}\, \boldsymbol{n}\, \mathrm{d}\mathcal{L}$$

5 Isoparametric mapping

Parent domain



Global domain



$$\mathbf{x}^e = \begin{bmatrix} x_1^e \\ x_2^e \\ x_3^e \\ x_4^e \end{bmatrix}, \mathbf{y}^e = \begin{bmatrix} y_1^e \\ y_2^e \\ y_3^e \\ y_4^e \end{bmatrix}$$

$$x = x(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\, \mathbf{x}^e \tag{9}$$

$$y = y(\xi, \eta) = \overline{\mathbf{N}}^e(\xi, \eta)\, \mathbf{y}^e \tag{10}$$

$$\begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix} = \mathbf{J} \begin{bmatrix} \mathrm{d}\xi \\ \mathrm{d}\eta \end{bmatrix}, \quad \mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial x}{\partial \eta} \\ \dfrac{\partial y}{\partial \xi} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{11}$$

$$\begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial x} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial y} \end{bmatrix} = \left(\mathbf{J}^{\mathrm{T}}\right)^{-1} \begin{bmatrix} \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \xi} \\ \dfrac{\partial \overline{\mathbf{N}}^e}{\partial \eta} \end{bmatrix} \tag{12}$$

6 Matrix inversion

The inverse of the matrix $\mathbf{M} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ is given by:

$$\mathbf{M}^{-1} = \frac{1}{\det(\mathbf{M})} \begin{bmatrix} M_{22} & -M_{12} \\ -M_{21} & M_{11} \end{bmatrix}, \quad \text{with } \det(\mathbf{M}) = M_{11}M_{22} - M_{12}M_{21}. \tag{13}$$

7 Stresses and strains

Hooke's generalised law: $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$

2D Strain-displ. relation: $\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial u_x}{\partial x} \\ \dfrac{\partial u_y}{\partial y} \\ \dfrac{\partial u_x}{\partial y} + \dfrac{\partial u_y}{\partial x} \end{bmatrix} = \tilde{\boldsymbol{\nabla}}\boldsymbol{u},\ \boldsymbol{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix},\ \tilde{\boldsymbol{\nabla}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$

Problem 1

a) To assemble the global stiffness matrix one can e.g. utilise the expanded versions of the element stiffness matrices

small mistakes on dof-element relation only ==> 0.5p
totally correct ==> 1.0p

$$\mathbb{K} = \mathbb{K}_1^{ee} + \mathbb{K}_2^{e} + \mathbb{K}_3^{e} =$$

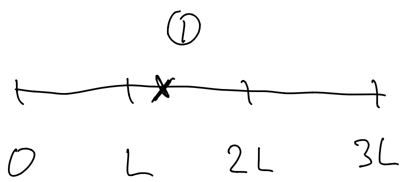
$$= \begin{bmatrix} a & 0 & 0 & -a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -a & 0 & 0 & a \end{bmatrix} + \begin{bmatrix} b & 0 & -b & 0 \\ 0 & 0 & 0 & 0 \\ -b & 0 & b & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & c & 0 & -c \\ 0 & 0 & 0 & 0 \\ 0 & -c & 0 & c \end{bmatrix}$$

$$= \begin{bmatrix} a+b & 0 & -b & -a \\ 0 & c & 0 & -c \\ -b & 0 & b & 0 \\ -a & -c & 0 & a+c \end{bmatrix} \quad ///$$

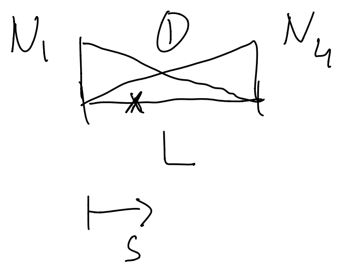
b)

$x = 4L/3 = L + \frac{L}{3}$ is within element 1.

Thus, only N_1 & N_4 are non-zero here



In local element coordinates (s) we have the shape functions as



$N_1 = 1 - \frac{s}{L} = \{ s = \frac{L}{3} \} = \frac{2}{3}$

$N_4 = \frac{s}{L} = \{ \quad \} = \frac{1}{3}$

N - 0.5p
u - 0.5p

$\Rightarrow \mathbb{N} = [\frac{2}{3}, 0, 0, \frac{1}{3}]$

Consequently, $u(x) = \mathbb{N} a = [\frac{2}{3}, 0, 0, \frac{1}{3}] \begin{bmatrix} 0.7 \\ 1.5 \\ 0.2 \\ 1.2 \end{bmatrix}$

$= \frac{2}{3} \cdot 0.7 + \frac{1}{3} 1.2 \approx 0.87.$ //

c) To see the effect of the spring, it is easier to look at the weak form.

clear motivation for 1.0p
if correct (in contrib and position but no motiv - 0.5p

$$\int_0^L \frac{dv}{dx} AE \frac{du}{dx}\, dx = \left[vAE \frac{du}{dx} \right]_0^L =$$

$$= \left[vA\sigma \right]_0^L = v(L)\,A(L)\,\sigma(L)$$

$$\underline{- v(0)\,A(0)\,\sigma(0)}$$

In this case, we have

$\sigma(0) = \dfrac{k\,u(0)}{A(0)}$ whereby the second term

in the RHS becomes

$- v(0)\, k\, u(0)$

With the FE-approx $u = \mathbb{N} a$
$v = \mathbb{N} c$ ← arbitrary

we get after some
elaboration this as

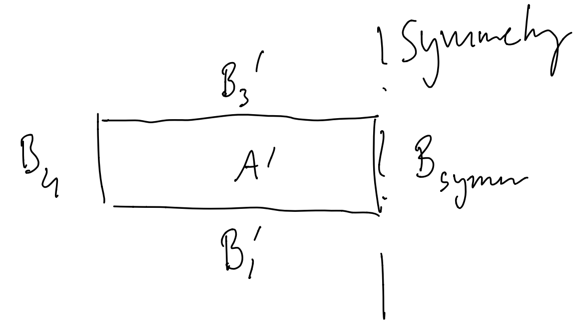
$$- c^T \underbrace{\mathbb{N}(0)^T k\, \mathbb{N}(0)}_{} a \quad \text{where } \mathbb{N}(0) = [0, 0, 1, 0]$$

$$\mathbb{K}_c = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then, the spring will give an extra contribution to the stiffness matrix $\mathbb{K}$ as $\mathbb{K}_{tot} = \mathbb{K} + \mathbb{K}_c$ but it has no direct influence on $\mathbb{f}$

Problem 2

Using symmetry, the problem is



$$-\mathrm{div}(\underline{q}) = \rho c_p \frac{\partial T}{\partial t} \quad \text{in } A' \text{ for } 0<t<t_{end}$$

$$T(x,y,t) = T_{ground} \text{ on } B_1'$$

$$q_n = \alpha(T - T_{air}) \text{ on } B_3' \text{ \& } B_4'$$

$$q_n = 0 \text{ on } B_{symm}$$

$$T(x,y,0) = T_{ground}$$

To get the weak form, we multiply by v & integrate over the domain (assume $t=1$ for simplicity)

$$-\int_{A'} v\,\mathrm{div}(\underline{q})\,dA = \int_{A'} v \rho c_p \frac{\partial T}{\partial t}\,dA$$

$$-\int_{A'} v \,\mathrm{div}(q_H)\, dA = \underbrace{-\int_{A'} \mathrm{div}(v q_H)\, dA}_{\underbrace{\int_{\mathcal{L}'} v q_H^T n \, d\mathcal{L}}_{\int_{\mathcal{L}'} v q_{fn} \mathcal{L}}} + \int_{A'} (\nabla v)^T q_H \, dA$$

$\uparrow$ boundary of A'

Adding all together, we get:

$$-\int_{\mathcal{L}'} v q_{fn} \, dA + \int_{A'} (\nabla v)^T q_H \, dA = \int_{A'} v \rho c_p \frac{\partial T}{\partial t} \, dA$$

$\Leftrightarrow \{ q_H = -k \nabla T \} \Rightarrow$

correct deriv - 0.5p
BC + IC - 0.5p

WEAK

$$\begin{cases} \int_{A'} v \rho c_p \frac{\partial T}{\partial t} \, dA + \int_{A'} (\nabla v)^T k \nabla T \, dA = -\int_{\mathcal{L}'} v q_{fn} \, dA = \\ T = T_{ground} \text{ on } B_1' \qquad = -\int_{B_1'} v q_{fn} \, dA - \int_{B_3' \cup B_4'} v \alpha (T - T_{air}) \, d\mathcal{L} \\ T(x,y,0) = T_{ground} \end{cases}$$

b) To arrive at the FE-form, we introduce the FE-approximation

$$T = \mathbb{N}\,\mathbf{a} \quad \text{with} \quad \mathbb{N} = [N_1(x,y) \;\; N_2(x,y) \; \dots \; N_{nno}(x,y)]$$

$$\Rightarrow \begin{cases} \dot{T} = \mathbb{N}\,\dot{\mathbf{a}} \\ \nabla T = \mathbb{B}\mathbf{a}, \; \mathbb{B} = \nabla \mathbb{N} \end{cases} \qquad \mathbf{a} = [T_1 \;\; T_2 \;\; T_3 \; \dots \; T_{nno}]^T$$

We also approximate v using Galerkin's method:

$$v = \mathbb{N}\mathbf{c} = \mathbf{c}^T \mathbb{N}^T \quad \text{with the same } \mathbb{N} \text{ & where } \mathbf{c} \text{ are arbitrary coefficients}$$

$$\nabla v = \mathbb{B}\mathbf{c}$$

$$(\nabla v)^T = \mathbf{c}^T \mathbb{B}$$

$$\mathbb{B} = \begin{bmatrix} \frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \dots & \frac{\partial N_{nno}}{\partial x} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & & \frac{\partial N_{nno}}{\partial y} \end{bmatrix}$$

Inserting the approximation yields:

$$\mathbf{c}^T \int_{A'} c_p \rho \, \mathbb{N}^T \mathbb{N} \, dA \, \dot{\mathbf{a}} + \mathbf{c}^T \int_{A'} \mathbb{B}^T k \mathbb{B} \, dA \, \mathbf{a} = -\mathbf{c}^T \int_{B_1'} \mathbb{N}^T q_n \, d\mathcal{L} - \mathbf{c}^T \int_{B_3' \cup B_4'} \alpha \mathbb{N}^T \mathbb{N} \, d\mathcal{L} \, \mathbf{a}$$

$$- \mathbf{c}^T \int_{B_3' \cup B_4'} \alpha \mathbb{N}^T T_{air} \, d\mathcal{L}$$

$$\iff$$

$$\mathbf{c}^T\Big[\underbrace{\int_{A'} \rho\, \mathbf{N}^T\mathbf{N}\,dA}_{\mathbf{C}_1}\,\dot{\mathbf{a}} + \Big(\underbrace{\int_{A'} \mathbf{B}^T k\, \mathbf{B}\,dA}_{\mathbf{K}} + \overbrace{\int_{B_3'\cup B_4'} \alpha\, \mathbf{N}^T\mathbf{N}\,d\mathcal{L}}^{\mathbf{K}_c}\Big)\mathbf{a}$$

$$+ \underbrace{\int_{B_1'} \mathbf{N}^T q_n\,d\mathcal{L}}_{-\mathbf{f}_g} + \underbrace{\int_{B_3'\cup B_4'} \alpha\, \mathbf{N}^T T_{air}\,d\mathcal{L}}_{-\mathbf{f}_c}\Big] = 0$$

Correct deriv - 0.5p

Since $\mathbf{c}$ arbitrary $\Rightarrow$

FE-form

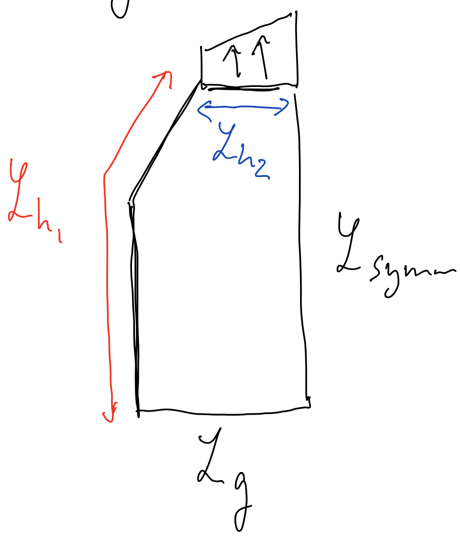
$$\begin{cases} \mathbf{C}_1\dot{\mathbf{a}} + (\mathbf{K}+\mathbf{K}_c)\mathbf{a} = \mathbf{f}_g + \mathbf{f}_c \\ T = T_{ground} \text{ on } B_1' \\ T(x,y,0) = T_{ground} \end{cases}$$

c) The results are not realistic. The temperature along B_1 should stay at T_{ground} due to the boundary conditions. Furthe[r] $T = 0^\circ C$ at $t = 0$. thereafter, the domain should be heated from the boundaries $B_2 - B_4$ but in the pictures it appears to be the opposite

correct answer and correct motiv 1.0p

Problem 3

For this case, one can utilise symmetry such that



Boundary conditions

$u_x = u_y = 0$ on $\mathcal{L}_g$

$\boldsymbol{t} = \boldsymbol{0}$ on $\mathcal{L}_{h_1}$

$\boldsymbol{t} = \begin{bmatrix} 0 \\ t_y(x) \end{bmatrix}$ on $\mathcal{L}_{h_2}$

$u_x = 0$, $t_y = 0$ on $\mathcal{L}_{symm}$

FE-form:

$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{a}$, $\boldsymbol{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & - & N_{nno} & 0 \\ 0 & N_1 & 0 & N_2 & & 0 & N_{nno} \end{bmatrix}$

$\Rightarrow \tilde{\nabla}\boldsymbol{u} = \underbrace{\tilde{\nabla}\boldsymbol{N}}_{\boldsymbol{B}}\boldsymbol{a}$

$\boldsymbol{a} = \begin{bmatrix} u_{x1} \\ u_{y1} \\ u_{x2} \\ u_{y2} \\ \vdots \\ u_{x\,nno} \\ u_{y\,nno} \end{bmatrix} \Leftarrow$ choice

$$\mathbb{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & -- & N_{nno} & 0 \\ 0 & N_1 & & 0 & N_{nno} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial x} & 0 & -- & \frac{\partial N_{nno}}{\partial x} & 0 \\ 0 & \frac{\partial N_1}{\partial y} & & 0 & \frac{\partial N_{nno}}{\partial y} \\ \frac{\partial N_1}{\partial y} & \frac{\partial N_1}{\partial x} & & \frac{\partial N_{nno}}{\partial y} & \frac{\partial N_{nno}}{\partial x} \end{bmatrix}$$

Using Galerkin's method to approximate v:

$v = \mathbb{N}\mathbb{c}$ where $\mathbb{c}$ contain arbitrary coeff.

$\tilde{\nabla} v = \mathbb{B}\mathbb{c}$, $(\tilde{\nabla} v)^T = \mathbb{c}^T \mathbb{B}^T$

Inserted in the weak form yields:

$$\mathbb{c}^T \int_A \mathbb{B}^T \mathbb{D} t \mathbb{B} \, dA \, a = \mathbb{c}^T \int_A \mathbb{N}^T \mathbb{b} t \, dA + \mathbb{c}^T \int_{\mathcal{L}_g} \mathbb{N}^T \overset{\text{unknown}}{\mathbb{h}} t \, d\mathcal{L} + \mathbb{c}^T \int_{\mathcal{L}_{h_1}} \mathbb{N}^T \mathbb{0} t \, d\mathcal{L}$$

$$+ \mathbb{c}^T \int_{\mathcal{L}_{h_2}} \mathbb{N}^T \begin{bmatrix} 0 \\ t_y \end{bmatrix} t \, d\mathcal{L}$$

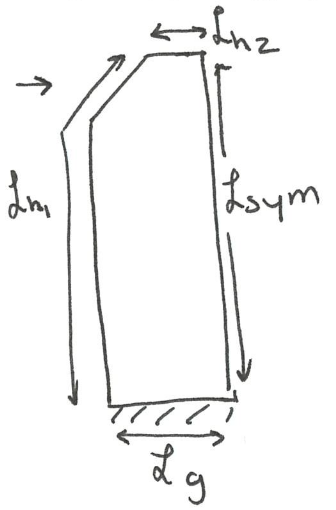
+ Lsymm term (see below)

$$\begin{cases} u_x = u_y = 0 & \text{on } \mathcal{L}_g \\ u_x = 0 & \text{on } L_{symm} \end{cases}$$

Correct deriv 0.5p
Correct BC - 0.5p

2018-04-06

Problem 3a)



FE-Form

$$\int_A \mathbf{B}^T \mathbf{D} \mathbf{B}\, dA\, \mathbf{a} \pm \int_A \mathbf{N}^T \mathbf{b}\, t\, dA + \int_{\mathcal{L}_g} \mathbf{N}^T \mathbf{t}\, t\, d\mathcal{L}$$

$$+ \int_{\mathcal{L}_{h_1}} \mathbf{N}^T \underbrace{\mathbf{t}}_{=\begin{bmatrix}0\\0\end{bmatrix}} t\, d\mathcal{L} + \int_{\mathcal{L}_{h_2}} \mathbf{N}^T \underbrace{\mathbf{t}}_{\begin{bmatrix}0\\t_y\end{bmatrix}} t\, d\mathcal{L}$$

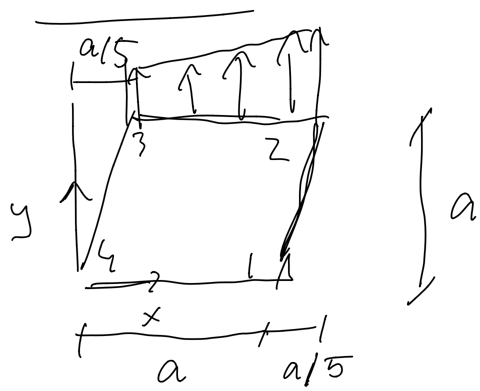
$$* \left\{ + \int_{\mathcal{L}_{sym}} \mathbf{N}^T \underbrace{\mathbf{t}}_{\begin{bmatrix}t_x\\0\end{bmatrix}} t\, d\mathcal{L} \right.$$

$u_x = u_y = 0$ on $\mathcal{L}_g$

$u_x = 0$ on $\underbrace{\mathcal{L}_{sym}}_{*}$

b) Element 1

Full point if correct with nodal positions (due to error in thesis). but not the combination



$x = N\hat{x}$, $\hat{x} = [a, 6a/5, a/5, 0]$ considering local coord syst as in the figure

$y = N\hat{y}$, $\hat{y} = [0, a, a, 0]$

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad N = [N_1(\xi,\eta) \; N_2(\xi,\eta) \; N_3(\xi,\eta) \; N_4(\xi,\eta)]$$

$$N_1 = \frac{1}{4}(\xi-1)(\eta-1), \quad \frac{\partial N_1}{\partial \xi} = \frac{1}{4}(\eta-1), \quad \frac{\partial N_1}{\partial \eta} = \frac{1}{4}(\xi-1)$$

$$N_2 = -\frac{1}{4}(\xi+1)(\eta-1), \quad \frac{\partial N_2}{\partial \xi} = -\frac{1}{4}(\eta-1), \quad \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(\xi+1)$$

$$N_3 = \frac{1}{4}(\xi+1)(\eta+1), \quad \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(\eta+1), \quad \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(\xi+1)$$

$$N_4 = -\frac{1}{4}(\xi-1)(\eta+1), \quad \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(\eta+1), \quad \frac{\partial N_4}{\partial \eta} = -\frac{1}{4}(\xi-1)$$

To get $J(\xi=1, \eta=0)$ we then have

$$\frac{\partial x}{\partial \xi}(\xi=1, \eta=0) = -\frac{1}{4}a + \frac{1}{4}\frac{6a}{5} + \frac{1}{4}\frac{a}{5} - \frac{1}{4}0 = \frac{a}{4}\left(-1 + \frac{6}{5} + \frac{1}{5}\right) = \frac{2a}{20}$$

$$= \frac{a}{10}$$

$$\frac{\partial x}{\partial \eta}(\xi=1,\eta=0) = 0\cdot a - \frac{1}{4}\cdot 2\cdot\frac{6a}{5} + \frac{1}{4}\cdot 2\cdot\frac{a}{5} + 0\cdot 0 =$$

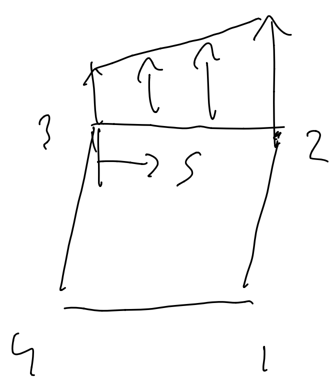
$$= \frac{a}{2}\left(-\frac{6}{5}+\frac{1}{5}\right) = -\frac{a}{2}$$

$$\frac{\partial y}{\partial \xi}(\xi=1,\eta=0) = \frac{1}{4}a + \frac{1}{4}a = \frac{a}{2}$$

$$\frac{\partial y}{\partial \eta}(\xi=1,\eta=0) = -\frac{2}{4}a + \frac{2}{4}a = 0$$

$$\Rightarrow \quad J = \begin{bmatrix} \frac{a}{10} & -\frac{a}{2} \\ \frac{a}{2} & 0 \end{bmatrix}, \quad \left(\det J = \frac{a^2}{4} \;..\; \text{ok!}\right)$$

c)



$$f_h^e = \int_{\overline{23}} N_e^T \begin{bmatrix} 0 \\ t_y \end{bmatrix} t \, d\mathcal{Y}$$

$$= \int_{\overline{23}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ N_2^e & 0 \\ 0 & N_2^e \\ N_3^e & 0 \\ 0 & N_3^e \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ t_y \end{bmatrix} t \, d\mathcal{Y}$$

integration - 0.5p
correct assemby of proper component - 0.5p

$$= \int_{\vec{23}} \begin{bmatrix} 0 \\ 0 \\ 0 \\ N_2^e t_y t \\ 0 \\ N_3^e t_y t \\ 0 \\ 0 \end{bmatrix} d\Gamma$$

$$\int_{\vec{23}} N_2^e t_y t \,\Gamma = \int_0^a \left(1 - \frac{s}{a}\right)\left(t_0 + k s\right) t \, ds \quad \text{with} \quad k = \frac{t_1 - t_2}{w/2}$$

$$= \int_0^a \left(t_0 + ks - \frac{t_0}{a} s - \frac{k}{a} s^2\right) t \, ds$$

$$= t\left[t_0 x + \left(k - \frac{t_0}{a}\right)\frac{s^2}{2} - \frac{k}{a}\frac{s^3}{3}\right]_0^a$$

$$= t\left(t_0 a + \frac{1}{2}\left(k - \frac{t_0}{a}\right) a^2 - \frac{k a^3}{3}\right)$$

$$= \frac{t t_0 a}{2} + \frac{t k}{6} a^2 = f_2^y$$

$$\int_{\vec{23}} N_3^e t_y t \, d\Gamma = \text{--} = \frac{t t_0 a}{2} + \frac{2}{6} t k a^2 = f_3^y$$

To assemble into the global load vector we utilise the numbering scheme

for node k: $u_{x,k} = u_{2k-1}$

$u_{y,k} = u_{2k}$

which means that f_2^y is added to component 4 in the global load vector & f_3^y to component 6 of the same.