VSM167 Finite element method - basics
Final exam 2018-01-12, 08:30-12:30

Instructor: Martin Fagerström (phone 070-224 8731). The instructor will visit the exam around 9:30 and 11:00.

Solution: Example solutions will be posted within a few days after the exam on the course homepage.

Grading: The grades will be reported to the registration office on Wednesday 31 January 2018 the latest.

Review: It will be possible to review the grading at the Division of Material and Computational Mechanics (floor 3 in M-building) on Monday 5 February 12:00-13:00 and Friday 9 February 12:00-13:00.

Permissible aids: No aids. **Note**: A formula sheet is appended to this exam thesis.

Problem 1

Consider one of the cylindrical concrete pillars supporting a bridge over a ravine. The concrete is to be considered as linear elastic with Young's modulus E [Pa]. The diameter of the pillar is D [m] and its total length is L [m] out of which a part of length h [m] is above the ground. The bottom of the pillar rests on rock which can be considered as rigid.

Due to the pressure exerted on the pillar from the ground, any downward movement of the pillar is resisted by an upward acting frictional force of magnitude $b_1(x) = \mu p(z(x))\pi D$ [N/m] where $p(z(x)) = p_0 \frac{z}{L-h}$ [Pa] (if $z > 0$, otherwise $p = 0$) is the ground pressure and $z(x) = x - h$ [m] is the distance below the ground surface. Furthermore, gravity yields a downwards action distributed force of magnitude $b_2(x) = \rho \frac{\pi D^2}{4} g$ [N/m] where ρ [kg/m^3] is the concrete density and g [m/s^2] is the gravity field. Finally, the bridge exerts loading on each pillar with a maximum force of P [N].

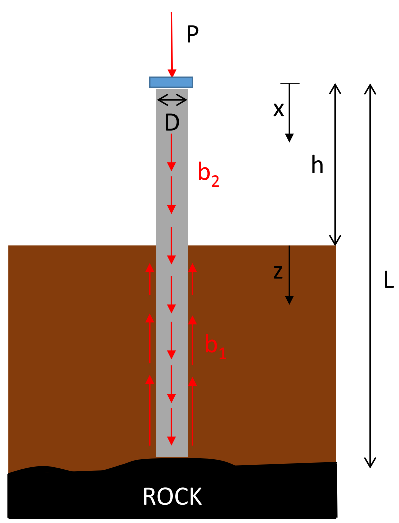


Figure 1: Sketch of the considered wall cross section.

A simple force balance of a short pillar segment of length dx yields that:

$$\frac{d\left(\sigma(x)A(x)\right)}{dx} + b(x) = 0$$

where $\sigma(x)$ [Pa] is the (uniaxial) stress in the pillar at coordinate x, $A(x)$ is the pillar cross section area and $b(x)$ [N/m] is the distributed force per unit length, positive in the $x-$direction. Furthermore, Hooke's law states that

$$\sigma(x) = E\varepsilon(x)$$

where

$$\varepsilon(x) = \frac{du(x)}{dx}$$

is the axial strain and $u(x)$ is the displacement of the pillar in a point at coordinate x.

Tasks on the next page

(a) Derive the strong form of the problem in terms of the primary unknown displacement field $u(x)$. You only need to specify one general expression for the governing equation for the whole pillar length $0 \leq x \leq L$ but please be careful to specify how the change in $b(x)$ affects the strong form (you only need to define $b(x)$ in terms of $b_1(x)$ and $b_2(x)$ - not more than that). **(0.5p)**

(b) Derive the corresponding weak form for the same problem. Please be very clear on how any possible natural (Neumann) boundary conditions enter in the governing equation. **(0.5p)**

(c) Derive the FE-form for the same problem using Galerkin's method. Be sure to carefully specify the contents (in general terms) of any matrices or vectors you introduce. **(0.5p)**

(d) Calculate the element contribution to the volume load vector $\boldsymbol{f}_l^e$ **for an element of length** L_e **in the interval** $0 \leq x \leq h$ if linear shape functions are used for the approximations. **(0.5p)**

(e) In the interval $h \leq x \leq L$, the calculation of the element contribution to the volume load vector $\boldsymbol{f}_l^e$ is more cumbersome due to b's dependence on x. In this interval, the element load contribution is more easily calculated using numerical integration. **Explain and motivate how many integration points that are necessary for an exact Gauss integration of this element load contribution** (again if linear shape functions are used for the approximations). **(0.5p)**

(f) Consider again the Gauss integration of the element load contribution in (e) for an element in the interval $h \leq x \leq L$ with nodal coordinates x_i and x_j ($x_j > x_i$). **What is/are the global coordinate(s) of the integration point(s) to be used**? **(0.5p)**

Problem 2

Consider the solid roof on a long building with triangular cross section in Figure 2, insulated at the bottom. The roof has a height $H = 5$ m and width $W = 10$ m and is made of an isotropic material which is considered to obey Fourier's law $\boldsymbol{q} = -k\boldsymbol{\nabla}T$ with heat conductivity k.

On a sunny winter day, the air temperature is $T_{air} = -5$ °C and the sun is shining on one of the roof sides (in red). The heat from the sun gives rise to a measured constant heat *influx* $h = 10$ W/m^2. In addition, the temperature at the bottom left corner is measured to be $\overline{T} = -4$ °C. Finally, the transfer coefficient between the roof material and air is α_{air}.

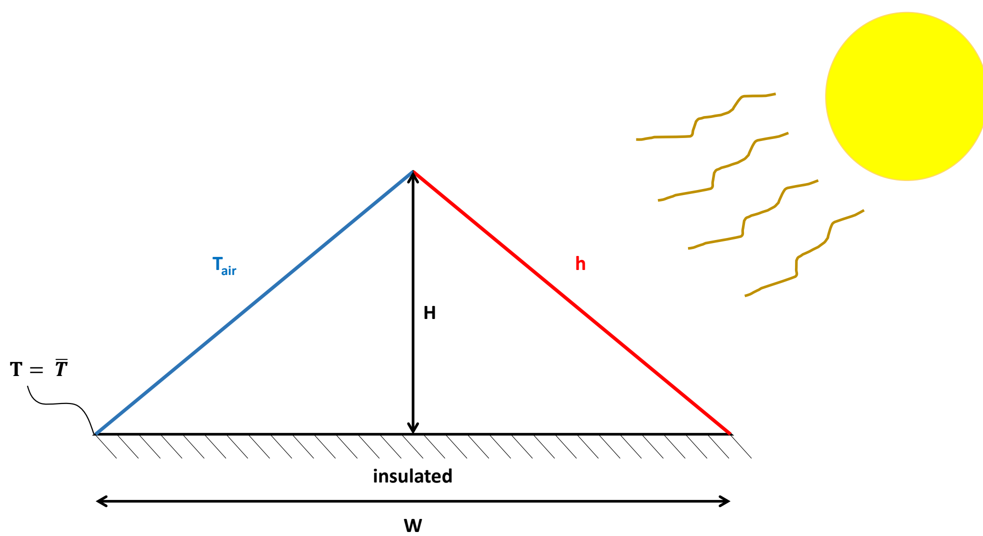


Figure 2: Sketch of the roof for Problem 2.

As the roof is long, the problem can be considered as two dimensional and the governing partial differential equation for the heat flow problem thereby becomes

$$\boldsymbol{\nabla}^{\mathrm{T}}\boldsymbol{q} \;=\; 0 \quad \text{in } \Omega$$

(a) Derive the weak form of the particular problem. Be specific in how the boundary conditions enter in the weak form. **(1.0p)**

Continued on the next page!

(b) By discretising the domain into two elements (see Figure 3) and introducing the linear finite element approximation on the form $T(x, y) \approx \boldsymbol{N}(x, y)\boldsymbol{a}$ and using Galerkin's method, the discrete form of the problem can be obtained as:

$$[\boldsymbol{K} + \boldsymbol{K}_c]\,\boldsymbol{a} = \boldsymbol{f}_b$$

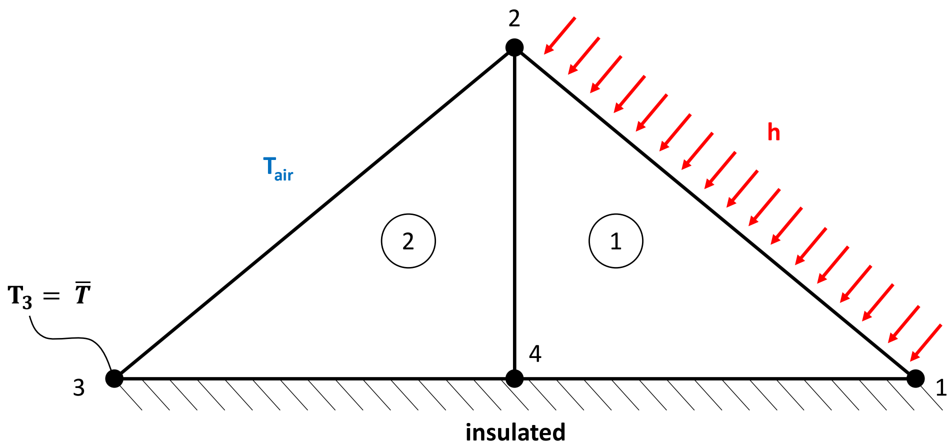


Figure 3: The roof domain in Problem 2 discretised into two elements.

Based on this, **calculate the element boundary load contribution $\boldsymbol{f}_b^e$ from element no 1** if the temperature is approximated with linear shape functions. Also **assemble this contribution into the global boundary load $\boldsymbol{f}_b$.** **(1.0p)**

(c) For the current problem discretisation (with two elements) the resulting discretised system of FE-equations will be of the form:

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

Partition the system into free and prescribed degrees of freedom and show how the problem should be solved when, as in this case, $T_3 = \overline{T}$. **(1.0p)**

Problem 3

Consider a 4-node iso-parametric bilinear element (as part of a given mesh, see also Figure 4) for linear elasticity in 2D with nodal coordinates

$$(x_1, y_1) = (0, 0), \quad (x_2, y_2) = (a, 0), \quad (x_3, y_3) = (a + c, b), \quad (x_4, y_4) = (c, b)$$

where a, b, c are given values. The parent element occupies the domain $-1 \leq \xi \leq 1$, $-1 \leq \eta \leq 1$ in the local coordinates (ξ, η).

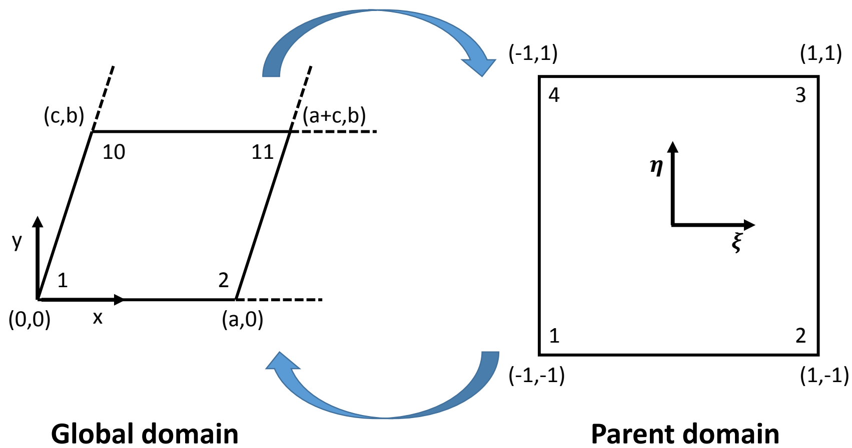


Figure 4: A 4-node iso-parametric bilinear element shown in the global domain and in the parent domain.

(a) In Figure 4, the global and local node numbers are indicated, respectively. **Propose a suitable numbering scheme to relate node numbers to numbers for the degrees-of-freedom for a 2D elasticity problem**. Then use this scheme to **indicate the global and local degrees-of-freedom in a sketch of the element in the global and parent domain**, respectively. The numbering of the global degrees-of-freedom must be such that each number can be used to identify the corresponding degree-of-freedom in the global displacement vector. **(0.5p)**

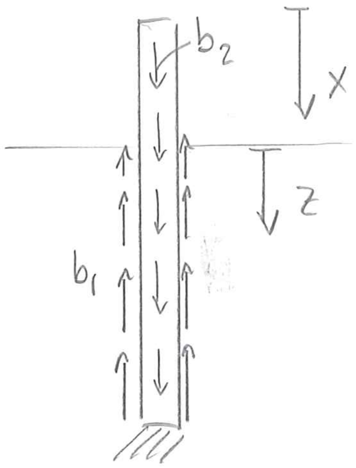
(b) Give the explicit expressions of the iso-parametric mappings $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ of the considered element. **(0.5p)**

(c) Compute the Jacobian matrix and its determinant in the centre point for the considered element. **(1p)**

(d) *Chance for extra points:* Can you check the value of the determinant of the Jacobian matrix to assess the correctness of your calculations in (c)?. **0.5p**

(e) Consider linear elasticity under plane strain conditions, for which the in-plane strain components are expressed as $\boldsymbol{\varepsilon} = \boldsymbol{B}^{\mathrm{e}} \boldsymbol{a}^{\mathrm{e}}$ in standard FE-notation. **Explain *first* without calculating any values the explicit contents of $\boldsymbol{B}^{\mathrm{e}}$**. Then, for the element centre point **calculate two non-zero components** in $\boldsymbol{B}^{\mathrm{e}}$ of your choice. **(1p)**

P1a)



$$\frac{d(\sigma(x)\,A(x))}{dx} + b(x) = 0$$

$$\Rightarrow \frac{d(E\,\varepsilon(x)\,A(x))}{dx} + b(x) = 0$$

$$\Rightarrow \frac{d}{dx}\left(E(x)\,A(x)\,\frac{du}{dx}\right) + b(x) = 0$$

E & A are constant →

Strong form

$$EA\,\frac{d^2u(x)}{dx^2} + b(x) = 0 \quad \forall\, 0 \le x \le L$$ 0.2p

$$b(x) = \begin{cases} b_2(x) & \text{if } x < h \\ b_2(x) - b_1(x) & \text{if } x > h \end{cases}$$ 0.1p

BCs:

$$\underbrace{A\,\sigma(0)}_{P(0)} = -P$$ 0.1p

$$u(L) = 0$$ 0.1p

b) multiply with weight function $v(x)$
& integrate over the domain

$$\underbrace{\int_0^L v(x) EA \frac{d^2u(x)}{dx^2} dx}_{} + \int_0^L v(x) b(x)\, dx = 0$$

$$\left[v(x) EA \frac{du}{dx}\right]_0^L - \int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx + \int_0^L v(x) b(x) dx = 0$$

$\Leftrightarrow$

$$\int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx = v(L) P(L) - v(0) P(0) + \int_0^L v(x) b(x) dx$$

0.3p

0.1p

$\Leftrightarrow$

weak form

$$\begin{cases} \int_0^L \frac{dv}{dx} EA \frac{du}{dx} dx = v(L) P(L) + v(0) P + \int_0^L v(x) b(x) dx \\ u(L) = 0 \end{cases}$$

0.1p

c) Introduce $u = \mathbb{N}\mathbf{a}$ with $\mathbb{N} = [N_1 \; N_2 \; \dots \; N_n]$

$$\mathbf{a} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

$\Rightarrow \frac{du}{dx} = \mathbb{B}\mathbf{a}$ with $\mathbb{B} = \left[\frac{dN_1}{dx} \; \frac{dN_2}{dx} \; \dots \; \frac{dN_n}{dx}\right]$

Use Galerkin's method $\Rightarrow$

$v = \mathbb{N}\mathbf{c} = \mathbf{c}^T\mathbb{N}^T$ & $\frac{dv}{dx} = \mathbb{B}\mathbf{c} = \mathbf{c}^T\mathbb{B}^T$, $\mathbf{c}$ - arbitr.

0.2p

Insert this in the weak form:

$$\int_0^L \mathbf{c}^T\mathbb{B}EA\,\mathbb{B}\mathbf{a}\,dx = \mathbf{c}^T\mathbb{N}^T(L)P(L) + \mathbf{c}^T\mathbb{N}^T(0)P + \int_0^L \mathbf{c}^T\mathbb{N}^T b(x)\,dx \quad \forall \mathbf{c}$$

Since $\mathbf{c}$ should be arbitrary, we have

$$\mathbf{c}^T\left(\mathbb{K}\mathbf{a} - \mathbf{f}_b - \mathbf{f}_l\right) = 0$$

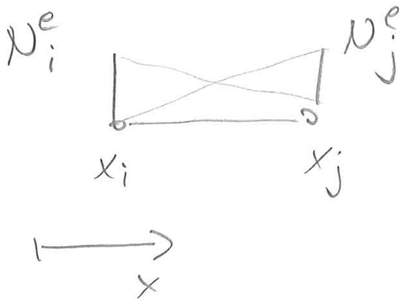
0.1p

$\Rightarrow$ FE-form
$$\begin{cases} \mathbb{K}\mathbf{a} = \mathbf{f}_b + \mathbf{f}_l \;\; \& \;\; u(L) = 0 \text{ with:} \\ \mathbb{K} = \int_0^L \mathbb{B}^T EA\,\mathbb{B}\,dx, \quad \mathbf{f}_b = \underbrace{\mathbb{N}^T(L)P(L)}_{\mathbf{f}_b^{(g)}} + \underbrace{\mathbb{N}^T(0)P}_{\mathbf{f}_b^{(h)}} \\ \mathbf{f}_l = \int_0^L \mathbb{N}^T b(x)\,dx \end{cases}$$

Res 0.2p

d/ $$f_l^e = \int_{x_i}^{x_j} N^{e^T} b(x)\,dx$$

in $0 \le x \le h$, $b(x) = b_2(x) = \rho \frac{\pi D^2}{4} g$



$$N^e = [N_i^e \quad N_j^e]$$

$$N_i^e = 1 - \frac{x - x_i}{x_j - x_i} = \frac{x_j - x_i - x + x_i}{x_j - x_i} = \frac{x_j - x}{x_j - x_i}$$

$$N_j^e = \frac{x - x_i}{x_j - x_i}$$

0.2p

$$f_l^e = \int_{x_i}^{x_j} \begin{bmatrix} N_i^e \\ N_j^e \end{bmatrix} \rho \frac{\pi D^2}{4} g\,dx = \rho \frac{\pi D^2}{4} g \cdot 1 \cdot \frac{(x_j - x_i)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{\pi \rho g D^2 (x_j - x_i)}{8} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [N]$$

ok

Rest 0.3p

e)

in $h \le x \le L$, $b(x) = b_2(x) - b_1(x)$ depends linearly on x since $b_1(x)$ does.

This means that

$$f_l^e = \int_0^L N^e(x)\, b(x)\, dx$$

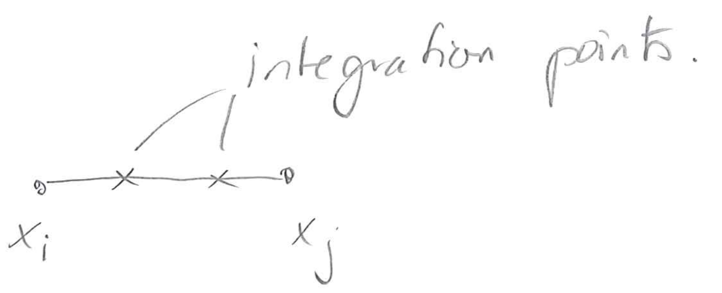
will mean to integrate two second order polynomials.

Gauss quadrature is accurate to the order $2n-1$ where n is the number of integration points.

Consequently, 2 integration points are needed per element since then a polynomial of order $2 \times 2 - 1 = 3$ will be integrated exactly.

0.5p if correct
otherwise 0p
No motivation => 0.25p

f)

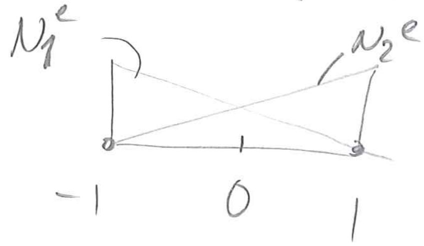


Gauss integration with two integration points mean that points should be placed at $\pm \frac{1}{\sqrt{3}}$ in the interval $-1 \le \xi \le 1$

0.1p

To find the global position of this we can utilise isoparametric mapping with linear shape functions:

$$x = N_1(\xi)\, x_1^e + N_2(\xi)\, x_2^e = N_1^e(\xi)\, x_i + N_2^e(\xi)\, x_j$$



$$N_1^e = (1-\xi)/2$$

$$N_2^e = (1+\xi)/2$$

The global coordinates of the integration points will then be:

int point 1, $\xi = -\frac{1}{\sqrt{3}}$; $x_1^{GP} = \left(1+\frac{1}{\sqrt{3}}\right)/2\, x_i + \left(1-\frac{1}{\sqrt{3}}\right)/2\, x_j$

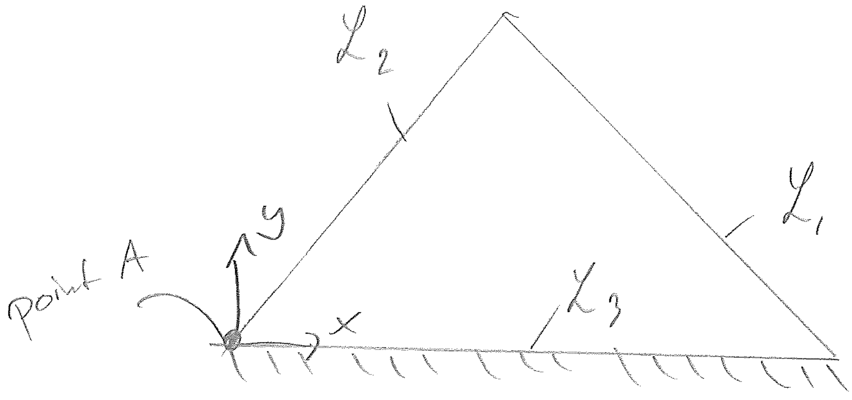
$$= \frac{x_i + x_j}{2} - \frac{(x_j - x_i)}{2\sqrt{3}} \quad // \quad 0.2p$$

int point 2, $\xi = \frac{1}{\sqrt{3}}$; $x_2^{GP} = \left(1-\frac{1}{\sqrt{3}}\right)/2\, x_i + \left(1+\frac{1}{\sqrt{3}}\right)/2\, x_j$

$$= \frac{x_i + x_j}{2} + \frac{(x_j - x_i)}{2\sqrt{3}} \quad // \quad 0.2p$$

P2

a)



Governing equation: $\nabla^T \underline{q} = 0$ in Ω

BC's:

$$q_n = -h \text{ on } L_1$$

$$q_n = \alpha(T - T_{air}) \text{ on } L_2$$

$$T(0,0) = \bar{T} \text{ (in point A)}$$

$$q_n = 0 \text{ on } L_3$$

To get the weak form, we multiply with a weight function $v(x,y)$ and integrate over the domain:

$$\int_A v \nabla^T \underline{q} \, dA = 0$$

$$\Leftrightarrow \int_A \nabla^T (v \underline{q}) \, dA - \int_A (\nabla^T v) \underline{q} \, dA = 0$$

$$\underline{q} = -k \nabla T \Rightarrow$$

$$\underbrace{\int_A \nabla^T(vq_f)\,dA}_{\oint_{\mathcal{L}} vq^T n\, d\mathcal{L}} + \int_A (\nabla^T v)\, k\, \nabla T\, dA = 0$$

$$\Rightarrow \oint_{\mathcal{L}} v q_{fn}\, d\mathcal{L} + \int_A (\nabla^T v)\, k\, \nabla T\, dA = 0$$

0.2p

$$\Leftrightarrow \int_A (\nabla^T v)\, k\, \nabla T\, dA = -\int_{\mathcal{L}_1} v(-h)\, d\mathcal{L} - \int_{\mathcal{L}_2} v\alpha(T - T_{air})\, d\mathcal{L}$$

$$- \int_{\mathcal{L}_3} v \cdot 0\, d\mathcal{L}$$

weak form

$$\Leftrightarrow \begin{cases} \int_A (\nabla^T v)\, k\, \nabla T\, dA = \int_{\mathcal{L}_1} v h\, d\mathcal{L} - \int_{\mathcal{L}_2} v\alpha(T - T_{air})\, d\mathcal{L} \\ T(0,0) = \bar{T} \end{cases}$$

$\Leftarrow$ 0.2p

Rest 0.6p

b)

Considering only the left hand side and introducing $T = N a$

$v = N c = c^T N^T$

implicitly using that c should be arbitrary means that

$$f_b = \int_{\mathcal{L}_1} N^T h \, d\mathcal{L} - \int_{\mathcal{L}_2} N^T \alpha (N a - T_{air}) \, d\mathcal{L}$$

Specifically for element 1 it has no part of its boundary along $\mathcal{L}_2$ which means that

0.2p

$$f_b^{e(1)} = \int_{\mathcal{L}_1} N^{e^T} h \, d\mathcal{L}$$

where, if we use the node numbering 1 - 2 - 4 we have

0.2p 0.2p

$$= \int_{\mathcal{L}_1} \begin{bmatrix} N_1^e \\ N_2^e \\ 0 \end{bmatrix} h \, d\mathcal{L} = \frac{h L_{12}^e}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

where L_{12}^e is the length between 1 & 2

$$L_{12}^e = \sqrt{\left(\frac{W}{2}\right)^2 + H^2} = \sqrt{50} \text{ m}$$

0.2p

with $h = 10 \ W/m^2$ we end up with

$$f_b^{e①} = \frac{10 \cdot \sqrt{50}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \ W/m$$

Assembling this into the global load vector f_b yields:

$$f_b = \begin{bmatrix} f_{b1} \\ f_{b2} \\ f_{b3} \\ f_{b4} \end{bmatrix} = \begin{bmatrix} 5\sqrt{50} \\ 5\sqrt{50} \\ 0 \\ 0 \end{bmatrix}$$

0.2p

not part of el. ①

no support along $\mathcal{L}_1$

c)

$$\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{bmatrix}$$

To solve the system, we partition the system into free dofs $a_f = \begin{bmatrix} T_1 \\ T_2 \\ T_4 \end{bmatrix}$ and prescribed dofs $a_p = [T_3]$.

0.2p

Rewriting the system of equations with the free dofs first yields:

$$\left[\begin{array}{ccc|c} K_{11} & K_{12} & K_{14} & K_{13} \\ K_{21} & K_{22} & K_{24} & K_{23} \\ K_{41} & K_{42} & K_{44} & K_{43} \\ \hline K_{31} & K_{32} & K_{34} & K_{33} \end{array}\right] \left[\begin{array}{c} T_1 \\ T_2 \\ T_4 \\ \hline T_3 \end{array}\right] = \left[\begin{array}{c} f_1 \\ f_2 \\ f_4 \\ \hline f_3 \end{array}\right]$$

$\mathbb{K}_{ff}$, $\mathbb{K}_{fp}$, a_f, f_f, $\mathbb{K}_{pf}$, $\mathbb{K}_{pp}$, a_p, f_p

0.6p

Or, in short:

$$\begin{bmatrix} \mathbb{K}_{ff} & \mathbb{K}_{fp} \\ \mathbb{K}_{pf} & \mathbb{K}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{a}_f \\ \mathbf{a}_p \end{bmatrix} = \begin{bmatrix} \mathbf{f}_f \\ \mathbf{f}_p \end{bmatrix}$$

This is then solved for T_1, T_2 & T_4 as

$$\mathbf{a}_f = \mathbb{K}_{ff}^{-1}\left(\mathbf{f}_f - \mathbb{K}_{fp}\mathbf{a}_p\right) \quad 0.2p$$

If desired, the reaction flux at node 3 can be post-processed in step 2 as

$$\mathbf{a}_p = \mathbb{K}_{pp}^{-1}\left(\mathbf{f}_p - \mathbb{K}_{pf}\mathbf{a}_f\right) \text{ for given } \mathbf{a}_f$$

Not needed

Problem 3a)

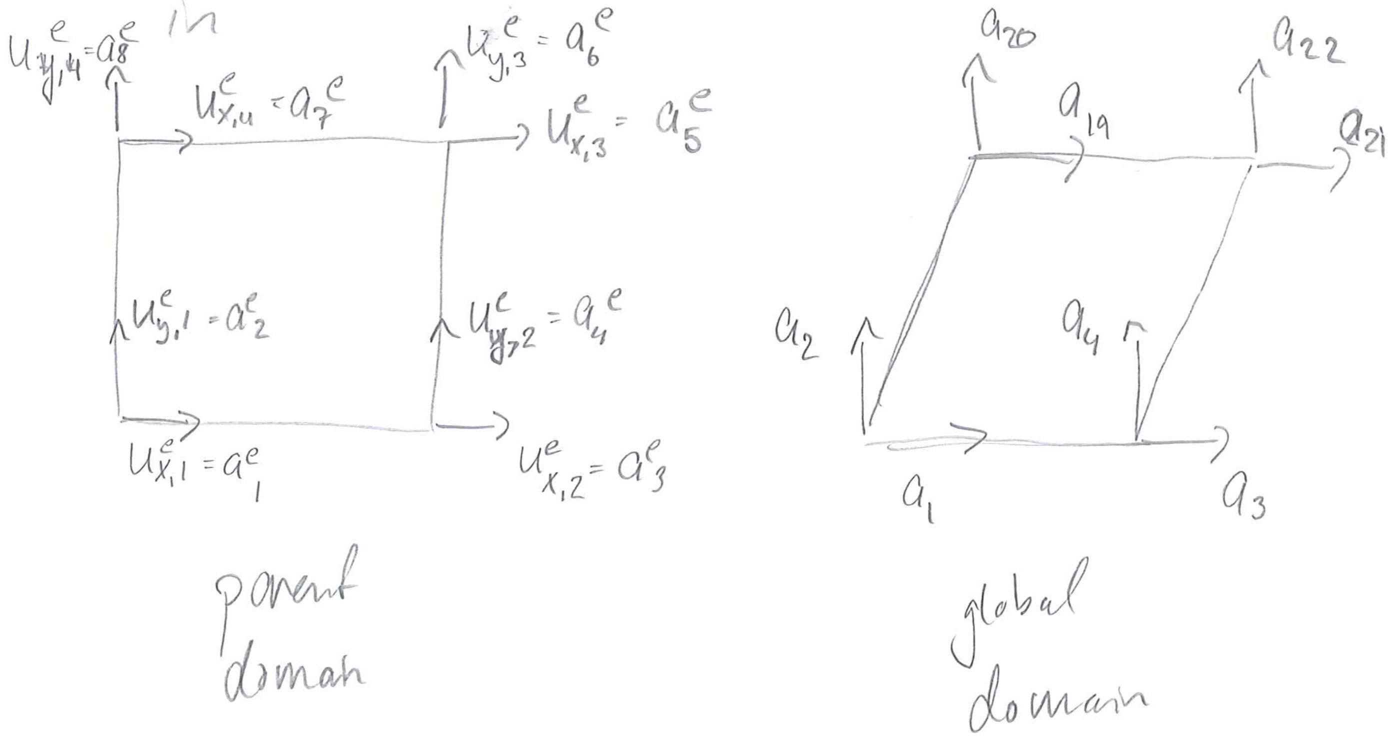
Suggested numbering scheme:

For each node k, we number the degrees of freedom as:

$$u_{x,k} = a_{2k-1}$$

$$u_{y,k} = a_{2k}$$

As an illustration, this can be shown in



0.5 if sketch consistent with numbering scheme

3b)

$$x(\xi,\eta) = \overline{N}^e(\xi,\eta)\,\hat{x}^e$$
$$y(\xi,\eta) = \overline{N}^e(\xi,\eta)\,\hat{y}^e$$

$$\overline{N}^e(\xi,\eta) = [N_1^e(\xi,\eta) \quad N_2^e(\xi,\eta) \quad N_3^e(\xi,\eta) \quad N_4^e(\xi,\eta)]$$

$$N_1^e = \frac{1}{4}(\xi-1)(\eta-1) \quad , N_2^e = -\frac{1}{4}(\xi+1)(\eta-1)$$

$$N_3^e = \frac{1}{4}(\xi+1)(\eta+1) \quad , N_4^e = -\frac{1}{4}(\xi-1)(\eta+1)$$

$$\hat{x}^e = \begin{bmatrix} 0 \\ a \\ a+c \\ c \end{bmatrix} \quad , \hat{y}^e = \begin{bmatrix} 0 \\ 0 \\ b \\ b \end{bmatrix}$$

/enough for 0.5p

$\Rightarrow$

$$x = -\frac{1}{4}(\xi+1)(\eta-1)a + \frac{1}{4}(\xi+1)(\eta+1)(a+c) - \frac{1}{4}(\xi-1)(\eta+1)c$$

$$y = \frac{1}{4}(\xi+1)(\eta+1)b - \frac{1}{4}(\xi-1)(\eta+1)b$$
$$\frac{1}{4}(\cancel{\xi\eta b} + \cancel{\xi b} + \eta b + b) - \frac{1}{4}(\cancel{\xi\eta b} + \cancel{\xi b} - \eta b - b) = \frac{1}{4}2\eta b + \frac{1}{4}2b$$

$$x = \frac{a}{2}\xi + \frac{c}{2}\eta + \left[\frac{a}{2} + \frac{c}{2}\right]$$

$$y = \frac{b}{2}\eta + \frac{b}{2}$$

3c)

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

$$\frac{\partial x}{\partial \xi} = \frac{a}{2}$$

$$\frac{\partial x}{\partial \eta} = \frac{c}{2}$$

$$\frac{\partial y}{\partial \xi} = 0$$

$$\frac{\partial y}{\partial \eta} = \frac{b}{2}$$

$$J(0,0) = \begin{bmatrix} \frac{a}{2} & \frac{c}{2} \\ 0 & \frac{b}{2} \end{bmatrix}$$ /0.5p

$$\det(J(0,0)) = \frac{ab}{4}$$ /0.5p

3d) det (J) represents the area scaling from parent (area 4) to global domain (area a·b)

Hence, we see directly that

$\det(\mathbb{J}) = \frac{a \cdot b}{4}$ is correct.

0.5p

3e)

$\sigma = \mathbb{B}^e a^e$ in an element

$$\mathbb{B}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & 0 & \frac{\partial N_2^e}{\partial x} & 0 & \cdots & \frac{\partial N_4^e}{\partial x} & 0 \\ 0 & \frac{\partial N_1^e}{\partial y} & 0 & \frac{\partial N_2^e}{\partial y} & & 0 & \frac{\partial N_4^e}{\partial y} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial y} & \frac{\partial N_2^e}{\partial x} & & \frac{\partial N_4^e}{\partial y} & \frac{\partial N_4^e}{\partial x} \end{bmatrix}$$

0.2

Thus, shape function derivatives need to be calculated. These can be obtained via:

0.2

$$\bar{\mathbb{B}}^e = \begin{bmatrix} \frac{\partial N_1^e}{\partial x} & \frac{\partial N_2^e}{\partial x} & \cdots & \frac{\partial N_4^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} & \frac{\partial N_2^e}{\partial y} & \cdots & \frac{\partial N_4^e}{\partial y} \end{bmatrix} = (\mathbb{J}^T)^{-1} \begin{bmatrix} \frac{\partial N_1^e}{\partial \xi} & \cdots & \frac{\partial N_4^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} & \cdots & \frac{\partial N_4^e}{\partial \eta} \end{bmatrix}$$

To get $\mathbb{B}^e$, one simply have to calculate $\bar{\mathbb{B}}^e$ & then extract components from there.

Calculating $\frac{\partial N_1^e}{\partial x}$ & $\frac{\partial N_1^e}{\partial y}$: & $\xi = 0, \eta = 0$

$$J^T(0,0) = \frac{1}{2}\begin{bmatrix} a & 0 \\ c & b \end{bmatrix}$$

$$(J^T)^{-1} = \frac{1}{2} \cdot \frac{4}{ab}\begin{bmatrix} b & 0 \\ -c & a \end{bmatrix}$$

$$= \frac{2}{ab}\begin{bmatrix} b & 0 \\ -c & a \end{bmatrix}$$

0.2

$$\begin{bmatrix} \frac{\partial N_1^e}{\partial x} \\ \frac{\partial N_1^e}{\partial y} \end{bmatrix} = (J^T)^{-1}\begin{bmatrix} \frac{\partial N_1^e}{\partial \xi} \\ \frac{\partial N_1^e}{\partial \eta} \end{bmatrix} = \frac{2}{ab}\begin{bmatrix} b & 0 \\ -c & a \end{bmatrix}\underbrace{\begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{4} \end{bmatrix}}_{\xi = \eta = 0}$$

$$= \begin{bmatrix} -\frac{1}{2a} \\ -\frac{1}{4}\left(\frac{-2c}{ab} + \frac{2a}{ab}\right) \end{bmatrix}$$

0.4