Problem 1.

Find the limit, if it exists, or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+2y^5}.$$

Solution

Approaching $(0,0)$ Along the line $x=0$: $\frac{xy}{x^2+2y^5} = \frac{0}{2y^5} = 0.$

Approaching $(0,0)$ along the line $y=x$:

$$\frac{xy}{x^2+2y^5} = \frac{x^2}{x^2+2x^5} = \frac{1}{1+2x^3} \longrightarrow 1 \neq 0$$

$\Rightarrow$ the limit does not exist.

Problem 2

Let $f(x,y) = x^2 + 2y^2 - 2x - 7$.

(a) Find and classify critical points of f

(b) Find the absolute maximum and minimum values of f on the region

$$D = \{(x,y) \mid x^2 + y^2 \le 4\}.$$

Solution

(a) $\nabla f = \langle 2x-2, 4y \rangle = 0$

$$2x - 2 = 0, \quad 4y = 0.$$

$$x = 1, \quad y = 0.$$

Critical point is $(1,0)$.

$f_{xx} = 2, \; f_{yy} = 4, \; f_{xy}^2 = 0.$

$D(1,0) = (f_{xx} f_{yy} - f_{xy})(1,0) = 2 \cdot 4 = 8 > 0$

$\Rightarrow (1,0)$ is a loc. min.

(b) ~~Abb.~~ Abs. max and min values must be either at critical pt $(1,0)$ or on the boundary of D which is the circle $x^2+y^2=4$.

At $(1,0)$ we have $f(1,0) = 1-2-7 = \boxed{-8.}$

To find abs. max and min at the boundary, we use Lagrange Method.

Let $g(x,y) = x^2+y^2-4$.

$\nabla g = \langle 2x, 2y \rangle$.

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = 0 \end{cases}$$

$$\begin{cases} \langle 2x-2, 4y \rangle = \lambda \langle 2x, 2y \rangle \\ x^2+y^2-4 = 0 \end{cases}$$

$$\begin{cases} 2x-2 = 2\lambda x & (1) \\ 4y = 2\lambda y & (2) \\ x^2+y^2 = 4 & (3) \end{cases}$$

$$\begin{cases} (1-\lambda)x = 2 & (1) \\ 2y = \lambda y & (2) \\ x^2+y^2 = 4 & (3) \end{cases}$$

From (2) we see that there are 2 cases

$y = 0$ ~~[illegible]~~ or $\lambda = 2$.

<u>case $y = 0$</u>: From (3) we find $\cancel{x}^2 = 4$

$x = \pm 2$.

From (1) we can find λ, but we don't need to know λ.

We obtained 2 pts: $(2, 0)$, $(-2, 0)$.

$f(2, 0) = \boxed{-7}$

$f(-2, 0) = \boxed{1.}$

<u>case $\cancel{\lambda} = 2$</u>: from (1) we find $x = \frac{1}{1-\lambda} = -1$.

from (3) $1 + y^2 = 4$, $y^2 = 3$, $y = \pm\sqrt{3}$

We get 2 pts: $(-1, \sqrt{3})$, $(-1, -\sqrt{3})$.

$f(-1, \sqrt{3}) = \boxed{2}$

$f(-1, -\sqrt{3}) = \boxed{2.}$

Thus $-8 = f(1, 0)$ ~~[illegible]~~ is abs. min,

$2 = f(-1, \sqrt{3}) = f(-1, -\sqrt{3})$ is abs. max.

Problem 4

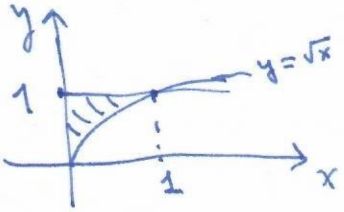
Compute the triple integral

$$\iiint_E y \, dV,$$

where E lies under the plane $z = 1 + 2x + y$ and above the region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 1$, $x = 0$.

Solution

$$\iiint_E y \, dV = \int_0^1 \int_{\sqrt{x}}^1 \int_0^{1+2x+y} y \, dz \, dy \, dx$$



$$= \int_0^1 \int_{\sqrt{x}}^1 \left[yz \right]_{z=0}^{z=1+2x+y} dy \, dx$$

$$= \int_0^1 \int_{\sqrt{x}}^1 y(1+2x+y) \, dy \, dx = \int_0^1 \int_{\sqrt{x}}^1 (y + 2xy + y^2) \, dy \, dx$$

$$= \int_0^1 \left[\frac{y^2}{2} + xy^2 + \frac{y^3}{3} \right]_{y=\sqrt{x}}^{y=1} dx = \int_0^1 \left(\frac{1}{2} + x + \frac{1}{3} - \frac{x}{2} - x^2 - \frac{x^{3/2}}{3} \right) dx$$

$$= \int_0^1 \left(\frac{5}{6} + \frac{x}{2} - x^2 - \frac{x^{3/2}}{3} \right) dx = \left[\frac{5}{6} x + \frac{x^2}{4} - \frac{x^3}{3} - \frac{2}{15} x^{5/2} \right]_{x=0}^{x=1}$$

$$= \frac{5}{6} + \frac{1}{4} - \frac{1}{3} - \frac{2}{15} = \frac{37}{60}.$$

Problem 5

Let C be the curve given by the parametrizatio

$$\bar{r}(t) = 2\cos t\,\bar{i} + (1-t)\,\bar{j} + 2\sin t\,\bar{k}, \quad 0 \le t \le \pi.$$

(a) Find the length of C

(b) Find the work done by the force field

$$\bar{F} = x\,\bar{i} + y^2\,\bar{j} + z\bar{k}$$

in moving a particle along C.

Solution (a) $\bar{r}'(t) = -2\sin t\,\bar{i} - \bar{j} + 2\cos t\,\bar{k}$

$$|\bar{r}'(t)| = \sqrt{4\sin^2 t + 1 + 4\cos^2 t} = \sqrt{5}$$

$$L = \int_0^{\pi} |\bar{r}'(t)|\,dt = \int_0^{\pi} \sqrt{5}\,dt = \pi\sqrt{5}.$$

(b) $$\text{work} = \int_C \bar{F}\cdot d\bar{r} = \int_C x\,dx + y^2\,dy + z\,dz$$

$$= \int_0^{\pi} \Big(2\cos t\,(-2\sin t) + (1-t)^2(-1) + 2\sin t\cdot 2\cos t\Big)\,dt$$

$$= \int_0^{\pi} \left(\cancel{-4\cos t\sin t} - (1-t)^2 + \cancel{4\sin t\cos t}\right)dt$$

$$= -\int_0^{\pi} (1-t)^2\,dt = \int_0^{\pi} (-1 + 2t - t^2)\,dt = \left[-t + t^2 - \frac{t^3}{3}\right]_0^{\pi}$$

$$= -\pi + \pi^2 - \frac{\pi^3}{3}.$$

Problem 6

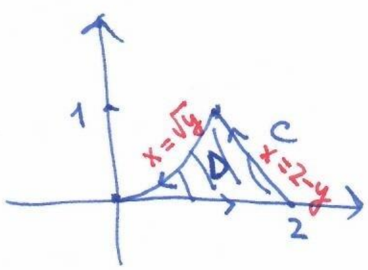
Compute the line integral

$$\int_C (x^3 \sin x - xy)\,dx + (2y - e^{y^3-5})\,dy,$$

where C is the curve consisting of the line segments going from $(0,0)$ to $(2,0)$ and from $(2,0)$ to $(1,1)$, and of the arc of the parabola $y = x^2$ from $(1,1)$ to $(0,0)$.

Solution

$$\int_C \underbrace{(x^3 \sin x - xy)}_{P}\,dx + \underbrace{(2y - e^{y^3-5})}_{Q}\,dy$$



$$\overset{\text{Green's Th.}}{=} \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D (0 - (-x))\,dA = \iint_D x\,dA = \int_0^1 \int_{\sqrt{y}}^{2-y} x\,dx\,dy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_{x=\sqrt{y}}^{x=2-y} dy = \frac{1}{2}\int_0^1 \left((2-y)^2 - y\right) dy = \frac{1}{2}\int_0^1 \left(4 - 4y + y^2 - y\right) dy$$

$$= \frac{1}{2}\int_0^1 \left(4 - 5y + y^2\right) dy = \frac{1}{2}\left[4y - \frac{5y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{y=1} = \frac{1}{2}\left(4 - \frac{5}{2} + \frac{1}{3}\right) = \frac{11}{12}.$$

Problem 7

Let E be the solid tetrahedron enclosed by the coordinate planes and the plane $2x+2y+z=2$. Let S be its surface given with the positive orientation. Let

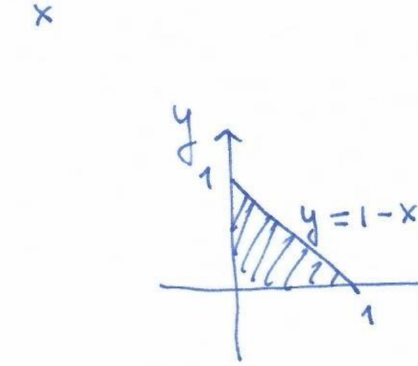
$$\bar{F} = y^2\bar{i} + xz\bar{j} + xz\bar{k}.$$

Find the flux of $\bar{F}$ across S, that is find

$$\iint_S \bar{F} \cdot d\bar{S}.$$

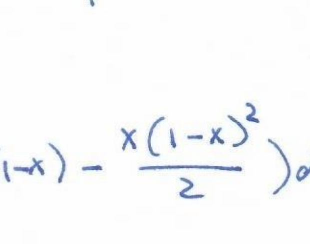
Solution

$\operatorname{div} \bar{F} = 0+0+x = x.$



$$\iint_S \bar{F} \cdot d\bar{S} = \iiint_E \operatorname{div} \bar{F} \, dV$$

$$= \iiint_E x \, dV = \int_0^1 \int_0^{1-x} \int_0^{2-2x-2y} x \, dz \, dy \, dx$$



$$= \int_0^1 \int_0^{1-x} x(2-2x-2y) \, dy \, dx$$

$$= 2\int_0^1 \int_0^{1-x} (x-x^2-xy) \, dy \, dx$$

$$= 2\int_0^1 \left[xy - x^2y - \frac{xy^2}{2} \right]_{y=0}^{y=1-x} dx = 2\int_0^1 \left(x(1-x) - x^2(1-x) - \frac{x(1-x)^2}{2} \right) dx$$

$$= 2\int_0^1 \left(x - x^2 \cancel{- x^2} + x^3 - \frac{x}{2} \cancel{+ x^2} - \frac{x^3}{2}\right) dx = 2\int_0^1 \left(\frac{x}{2} - x^2 + \frac{x^3}{2}\right) dx$$

$$= \int_0^1 \left(x - 2x^2 + x^3\right) dx = \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_{x=0}^{x=1} = \frac{1}{2} - \frac{2}{3} + \frac{1}{4} =$$

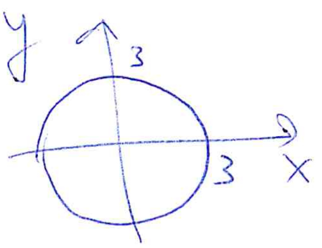
Problem 3

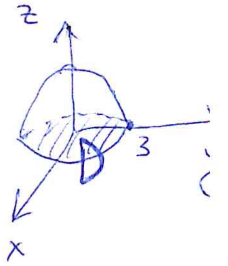
We find the intersection of the paraboloid $z = 9 - x^2 - y^2$ with the xy-plane:

$$0 = 9 - x^2 - y^2.$$

$$x^2 + y^2 = 9$$

It is a circle $\longrightarrow$





Therefore we need to find the volume under the graph of the f-n $9 - x^2 - y^2$ above the disc. $D = \{(x,y) \mid x^2 + y^2 \le 9\}$

$$\text{Volume} = \iint_D (9 - x^2 - y^2)\, dA \qquad \underline{\underline{\text{use polar coordinates}}}$$

$$= \int_0^{2\pi} \int_0^3 (9 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^3 (9r - r^3)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4} \right]_{r=0}^{r=3} d\theta = \int_0^{2\pi} \frac{81}{4}\, d\theta = \frac{81}{4} \cdot 2\pi = \frac{81\pi}{2}.$$

$\square$.