Tentamen MVE255. Mathematical sciences, Chalmers University of Technology

Datum:	August 19th 2021, 14.00
Allowed aids:	Any.
Grade limits:	20 points for the grade 3. 30 points for the grade 4 40 points for the grade 5.

There are in total 50 points to collect.

Calculations and arguments should be presented in full. **Only providing an answer will normally not be rewarded with points.** Solutions may be written in Swedish or English.

If you use any external tool or resource, you should reference it. See https://chalmers.instructure.com/courses/10111/pages/reference-rules the Canvas Page **for more information. Not referencing properly may result in point deduction.**

The exam consists of **eight (8)** problems. They are distributed over **three (3)** sheets.

Good luck!

Problem 1 (6P)

Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be defined through

$$g(x, y, z) = \begin{bmatrix} yx \\ xz \\ zy \end{bmatrix}.$$

(1) Find the derivative of g. (4p)

(2) Evaluate g and g' in $\langle 1, 1, 1\rangle$. (2p)

Problem 2 (6P)

Calculate $\iint_A y \, \mathrm{d}x\mathrm{d}y$, where A is the part of the square $[0, 1]^2$ that lies above the curve $y = x^2$.

Problem 3 (7P)

Let D be the part of the cone $K = \{\langle x, y, z\rangle, | \, x^2 + y^2 \leq z^2\}$ which lies inside the cylinder $Z = \{\langle x, y\rangle, | \, x^2 + y^2 = 1, 0 \leq z \leq 4\}$. Determine the center of mass of D.

Tip: You may use the symmetries of the set to your advantage.

Problem 4 (7P)

Let the function f be given through

$$f(x, y) = x^2 - xy + y^2 - 2x + y + 1.$$

(1) Find and classify all critical points of the function on $\mathbb{R}^2$. (4p)

(2) The function has a maximum on the set (this does not need to be shown)

$$G = \{\langle x, y\rangle \, | \, x^4 + y^4 \leq 1\}.$$

Is this maximum attained in the point $\langle -1/\sqrt[4]{2}, 1/\sqrt[4]{2}\rangle$? (3p)

Problem 5 (6P)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a function with

$$f'(x, y) = \begin{bmatrix} ye^{-x} & -e^{-x} \\ 2xe^{-y} & -x^2e^{-y} \end{bmatrix}$$

and $g : \mathbb{R}^3 \to \mathbb{R}^2$ the function

$$g(s, t, u) = \begin{bmatrix} st \\ ut \end{bmatrix}$$

Determine the value of $(f \circ g)'(0, 0, 1)$.

Problem 6 (6P)

Let $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ be defined through

$$\mathbf{v}(x, y, z) = \begin{bmatrix} ye^{xz}(1 + xz) \\ xe^{xz} \\ x^2ye^{xz} \end{bmatrix}.$$

(1) Show that $\mathbf{v}$ is conservative, and determine a potential for $\mathbf{v}$ (3p)

(2) Calculate the value of the line integral

$$\int_\gamma \mathbf{v} \cdot \mathrm{d}\mathbf{r},$$

where γ is the curve (3p)

$$\gamma(t) =< t\cos(t), t\sin(t), t >, \; t \in [0, 2\pi].$$

Problem 7 (6P)

Let γ be the closed and non-self-intersecting curve given by the parametrization

$$\gamma(t) = \begin{bmatrix} (t+1)\sin(t) \\ -\cos(t) \\ \cos(t) - (t+1)\sin(t) + 1 \end{bmatrix}, t \in [0, 2\pi]$$

and $\mathbf{v}$ the vector field

$$\mathbf{v}(x, y, z) = (x + y + z) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(1) Show that all points in γ lie in the plane $x + y + z = 1$, and thus that γ is the boundary curve of a region in the same plane. (1p)

(2) Calculate the curl of $\mathbf{v}$. What is the dot product between $\operatorname{curl}\mathbf{v}$ and the normal of the plane $x + y + z = 1$? (2p)

(3) Show that the line integral of $\mathbf{v}$ along γ is zero. (3p)

Problem 8 (6P)

Let $\mathbf{w} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined through

$$\mathbf{w}(x, y, z) = \begin{bmatrix} xy^2 \\ yz^2 \\ zx^2 \end{bmatrix}.$$

Let furthermore $K \subseteq \mathbb{R}^3$ be a solid which is a subset of the unit ball, i.e. $|\mathbf{p}| < 1$ for all $\mathbf{p} \in K$ and has non-empty interior. Show that

$$0 < \int_{\partial K} \mathbf{w} \cdot \mathrm{d}\mathbf{S} \leq \frac{4\pi}{5},$$

where the surface element is pointing out of K.

Tip: Apply the divergence (Gauß) theorem.