Exam LMA017

Mathematical sciences, Chalmers University of Technology

Datum:	January 4th 2021, 08:30
Examiner:	Axel Flinth
Allowed aids:	Any.
Grade limits:	20 points for the grade 3. 30 points for the grade 4 40 points for the grade 5.

There are in total 50 points to collect.

Calculations and arguments should be presented in full. **Only providing an answer will normally not be rewarded with points.** Solutions may be written in Swedish or English (or German).

If you use any external tool or resource, you should reference it. See the Canvas Page for more information. Not referencing properly may result in point deduction.

The exam consists of **eight (8)** problems. They are distributed over **four (4)** pages.

Good luck!

SOLUTIONS

No guarantee of correctness!

Problem 1 (6p)

Let $g : \mathbb{R}^2 \to \mathbb{R}^3$ be defined through

$$g(x, y) = \begin{bmatrix} x^2y \\ y\sin(x) \\ x \end{bmatrix}.$$

1. Calculate the derivative of g. (3p)
2. Evaluate g and g' in $\langle 1, 0\rangle$. (3p)

Solution. (a) The derivative is given by the matrix formed when writing the two partial derivatives next to each other

$$g'(x,y,z) = \begin{bmatrix} \frac{\partial g}{\partial x}(x,y,z) & \frac{\partial g}{\partial y}(x,y,z) \end{bmatrix}.$$

Thus

$$g'(x,y) = \begin{bmatrix} 2xy & x^2 \\ y\cos(x) & \sin(x) \\ 1 & 0 \end{bmatrix},$$

(b) We now simply need to insert $x = 1$ and $y = 0$. We get

$$g(1,0) = \begin{bmatrix} 1^2 \cdot 0 \\ 0 \cdot \sin(1) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$g'(1,0,1) = \begin{bmatrix} 2 \cdot 1 \cdot 0 & 1^2 \\ 0\cos(1) & \sin(1) \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & \sin(1) \\ 1 & 0 \end{bmatrix}.$$

□

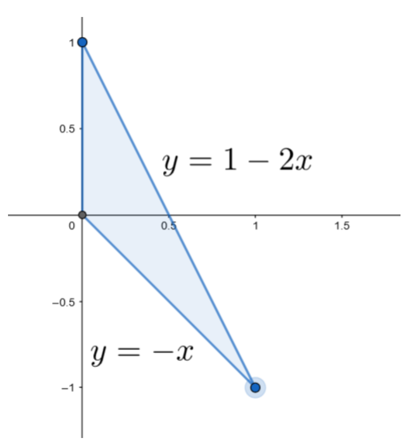
Problem 2 **(7p)**

Consider a triangle with corners $\langle 0,0\rangle$, $\langle 0,1\rangle$ and $\langle 1,-1\rangle$. Calculate the center of mass of the triangle

Solution. Let T be the triangle. We need to calculate the expressions

$$p_x = \frac{\int_T x \,\mathrm{d}x\mathrm{d}y}{\int_T \mathrm{d}x\mathrm{d}y} \quad p_y = \frac{\int_T y \,\mathrm{d}x\mathrm{d}y}{\int_T \mathrm{d}x\mathrm{d}y}.$$

(note that when we speak about the center of mass of a geometrical object per se, ρ is assumed constant, and WLOG 1).



Let us first note that $\int_T \rho(x,y)\,\mathrm{d}x\mathrm{d}y$ is equal to the area of the triangle, which can be calculated with the help of the formula $\frac{1}{2}$base $\cdot$ height:

$$\int_T \rho(x,y)\,\mathrm{d}x\mathrm{d}y = \frac{1}{2}\cdot 1\cdot 1 = \frac{1}{2}.$$

In order to evaluate the other integrals, we need to find a description of T as a simple domain. To do this, the simplest thing to do is that for each $x \in [0,1]$, the y-coordinates of the points range from $-x$ to $1-2x$. Thus, for any function g

$$\int_T g\,\mathrm{d}x\mathrm{d}y = \int_0^1 \int_{-x}^{1-2x} g(x,y)\,\mathrm{d}y\mathrm{d}x.$$

We now use this to calculate

$$\begin{aligned}
\int_T x\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_{-x}^{1-2x} x\,\mathrm{d}y\mathrm{d}x = \int_0^1 x(1-x)\,\mathrm{d}x = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6} \\
\int_T y\,\mathrm{d}x\mathrm{d}y &= \int_0^1 \int_{-x}^{1-2x} y\,\mathrm{d}y\mathrm{d}x = \int_0^1 \left[\frac{y^2}{2}\right]_{y=-x}^{1-2x} \mathrm{d}x = \int_0^1 \frac{(1-4x+4x^2)-x^2}{2}\,\mathrm{d}x \\
&= \left[\frac{x}{2} - x^2 + \frac{x^3}{2}\right]_{x=0}^1 = 0
\end{aligned}$$

The center of mass of the triangle is thus

$$p_x = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}, \quad p_y = \frac{0}{\frac{1}{6}} = 0.$$

□

Problem 3 **(6p)**

Calculate $\iiint_S z\mathrm{d}x\mathrm{d}y\mathrm{d}z$, where S is the part of the unit ball that lies above the xy-plane.

Solution. The easiest technique to solve this integral is probably to transform to spherical coordinates $\langle \rho\sin(\phi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), \rho\cos(\varphi)\rangle$. The unit ball is described via the inequalities

$$0 \le \rho \le 1,\ 0 \le \varphi \le \pi,\ 0 \le \theta \le 2\pi.$$

Since we only want to integrate over the part of the ball which lies above the xy-plane, we need to restrict the domain for φ to $[0, \frac{\pi}{2}]$. The functional determinant of the spherical coordinates is $\rho^2\sin(\varphi)$, and $z = \rho\cos(\varphi)$. Thus

$$\iiint_S z\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho\cos(\varphi)\cdot\rho^2\sin(\varphi)\,\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\rho.$$

It now only remains to evaluate the integral. (This can surely be done via a computer algebra system)

$$\begin{aligned}\int_0^1 \int_0^{\pi/2} \int_0^{2\pi} \rho^3 \cos(\varphi)\sin(\varphi)\,\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}\rho &= 2\pi \int_0^1 \int_0^{\pi/2} \frac{\rho^3 \sin(2\varphi)}{2}\,\mathrm{d}\varphi\mathrm{d}\rho \\ &= 2\pi \int_0^1 \left[\frac{-\rho^3\cos(2\varphi)}{4}\right]_{\varphi=0}^{\varphi=\pi/2} \mathrm{d}\rho = \frac{\pi}{2}\int_0^1 2\rho^3\,\mathrm{d}\rho = \frac{\pi}{4}\end{aligned}$$

□

Problem 4 **(6p)**

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined through

$$f(x,y) = \cos\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}y\right).$$

Determine the global maximum and minimal values of f on the square $[-1,1]^2$.

Solution. $[-1,1]^2$ is a compact domain and f is continuous. Thus, f has maximal and minimal values, and we can find them among the critical points (on the boundary and in the interior).

Let us begin with the boundary. The boundary is given by the points of the form $\langle \pm 1, t\rangle$ and $\langle s, \pm 1\rangle$ for $|s| \leq 1$ and $|t| \leq 1$. In all of these points, since $\cos\left(\frac{\pi}{2}\right) = \cos\left(-\frac{\pi}{2}\right) = 0$, f is equal to 0. The function is thus constant on the boundary, and therefore all points on the boundary are candidates for the extrema.

We move on to search for critical points in the interior. We have

$$\nabla f(x,y) = \frac{\pi}{2}\begin{bmatrix} -\sin\left(\frac{\pi}{2}x\right)\cos\left(\frac{\pi}{2}y\right) \\ -\cos\left(\frac{\pi}{2}x\right)\sin\left(\frac{\pi}{2}y\right)\end{bmatrix}.$$

This is zero if and only if $\sin\left(\frac{\pi}{2}x\right) = \sin\left(\frac{\pi}{2}y\right) = 0$, since cos is larger than zero on $(-\pi/2, \pi/2)$. Thus, the only critical point in the interior is given by $\langle 0, 0\rangle$.

We now compare the function values. As discussed, the function is constantly equal to 0 on the boundary, and $f(0,0) = 1$. Thus, the maximal value is 1, and the minimal value is 0.

□

Problem 5 **(7p)**

Milk is poured into a mug of coffee. The amount of coffee lies between 2.3 and 2.7 deciliters, and the amount of milk is somewhere between 0.3 and 0.7 deciliters. Use the Schrankensatz to give an estimate of the fraction of the liquid in the cup that is milk, with error bounds.

Solution. Let c be the amount of coffee in the cup, and m the amount of milk. From the text, we can infer that $c = 2.5 \pm 0.2$ and $m = 0.5 \pm 0.2$. The entity we want to estimate is the fraction of milk, which is described by

$$f : [0, \infty[^2 \to \mathbb{R}, (c, m) \mapsto \frac{m}{c+m}.$$

The Schrankensatz says

$$|f(c,m) - f(2.5, 0.5)| \leq M_c\,|c - 2.5| + M_m\,|m - 0.5|\,,$$

where

$$M_c = \sup_{\substack{|c-2.5|\leq 0.2\\ |m-0.5|\leq 0.2}} \left|\frac{\partial f}{\partial c}(c,m)\right|$$
$$M_m = \sup_{\substack{|c-2.5|\leq 0.2\\ |m-0.5|\leq 0.2}} \left|\frac{\partial f}{\partial m}(c,m)\right|.$$

We estimate

$$M_c = \sup_{\substack{|c-2.5|\leq 0.2\\ |m-0.5|\leq 0.2}} \left|-\frac{m}{(c+m)^2}\right| \leq \frac{0.7}{(2.3+0.3)^2}$$
$$M_m = \sup_{\substack{|c-2.5|\leq 0.2\\ |m-0.5|\leq 0.2}} \left|-\frac{c}{(c+m)^2}\right| \leq \frac{2.7}{(2.3+0.3)^2}.$$

This gives the final bound

$$M_c\,|c-2.5| + M_m\,|m-0.5| \leq 0.2 \cdot \frac{0.7}{(2.3+0.3)^2} + 0.2 \cdot \frac{2.7}{(2.3+0.3)^2} \leq 0.101.$$

Since $f(2.5, 0.5) = \frac{1}{6} \approx 0.167$, we obtain

$$f(c,m) \approx 0.167 \pm 0.101.$$

$\square$

Problem 6 **(6p)**

Let $\mathbf{w} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\mathbf{w}(x,y,z) = \begin{bmatrix} xy^2 + z \\ yz^2 + x \\ zx^2 + y \end{bmatrix}.$$

Let further K be the cube $[0,1]^3 = \{|x,y,z\,|\,0 \le x,y,z \le 1\}$, and S its surface. Calculate the flow of $\mathbf{w}$ out ouf K, i.e.

$$\iint_S \mathbf{w} \cdot \mathrm{d}\mathbf{S}.$$

Solution. We can use Gauß's theorem (the divergence theorem)

$$\iint_S \mathbf{w} \cdot \mathrm{d}\mathbf{S} = \iint_K \mathrm{div}\mathbf{w}\mathrm{d}V.$$

We have

$$\mathrm{div}\mathbf{w} = \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z} = y^2 + z^2 + x^2.$$

The integral that we need to evaluate is thus

$$\iint_K (x^2 + y^2 + z^2)\,\mathrm{d}V.$$

We do not need to invest much work in determining the limits for this integral - they are equal to 0 and 1 for each variable.

$$\begin{aligned}\int_0^1 \int_0^1 \int_0^1 (x^2+y^2+z^2)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 \int_0^1 \frac{1}{3} + y^2 + z^2\,\mathrm{d}y\mathrm{d}z = \int_0^1 \frac{1}{3} + \frac{1}{3} + z^2\,\mathrm{d}z \\ &= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1.\end{aligned}$$

□

Problem 7 **(6p)**

A ball is rolling on the graph of a twice differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$. The $\langle x,y\rangle$-coordinates of the ball follow a curve $\gamma(t)$. At $t = 0$, these coordinates reside in the point $\gamma(0) = \langle 0,0\rangle$, and the velocity of them is $\gamma'(0) = \langle 1,2\rangle$. We know that

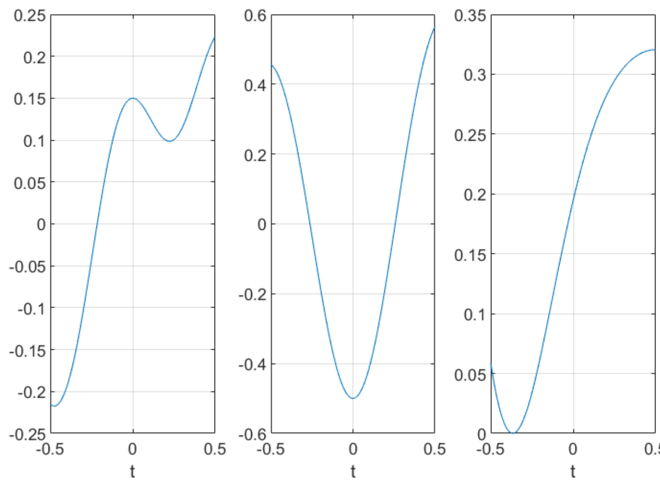
$$\nabla f(0,0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad f''(0,0) = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

We let g denote the function which describes how the height of the ball depends on time, i.e. $g = f \circ \gamma$.

(a) What is the derivative of g in 0, i.e. $g'(0)$? (2p)

(b) What is the second derivative of g in 0, i.e. $g''(0)$? (2p)

(c) Which of the following graphs depicts g? (2p)



Only ruling out one graph with a good justification will be awarded one point.

Solution. (a) By the chain rule, we have

$$g'(\theta) = f'(\gamma(\theta))\gamma'(theta) = \nabla f(\gamma(\theta)) \cdot \gamma'(\theta).$$

Evaluating this in 0, we have

$$g'(0) = \nabla f(\gamma(0)) \cdot \gamma'(0) = \nabla f(0,0) \cdot \gamma'(0) = 0,$$

since $\nabla f(0,0) = \mathbf{0}$.

(b) We start by using the product rule for vector-valued functions to determine

$$g''(\theta) = (\nabla f \circ \gamma)'(\theta) \cdot \gamma'(\theta) + \nabla f(\gamma(\theta)) \cdot \gamma''(\theta).$$

We know apply the chain rule for a second time see that

$$(\nabla f \circ \gamma)'(\theta) = (\nabla f)'(\gamma(\theta))\gamma'(\theta) = f''(\gamma(\theta))\gamma'(\theta).$$

Thus

$$g''(\theta) = (f''(\gamma(\theta))\gamma'(\theta)) \cdot \gamma'(\theta) + \nabla f(\gamma(\theta)) \cdot \gamma''(\theta).$$

We now need to evaluate this in $\theta = 0$. Since $\nabla f(\gamma(0)) = \mathbf{0}$, the second term vanishes. The expression is thus equal to

$$g''(0) = (f''(\mathbf{0})\gamma'(0)) \cdot \gamma'(0) = \left(\begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} = -3.$$

(c) g is a function that has a derivative equal to 0 in $\theta = 0$. The rightmost graph is thus out of question. To determine whether the left or the middle one is the right choice, we notice that the second derivative is negative. Thus, the graph locally looks like a downward facing parabola, and it must be the **leftmost** graph that describes the g. □

Problem 8 **(6p)**

Let $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\mathbf{v}(x, y) = \begin{bmatrix} -y \\ x \end{bmatrix}$$

(a) Determine the line integral of $\mathbf{v}$ along the unit circle. You can choose which direction the circle is traversed. (2p)

(b) Does there exist a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ with the property that

$$f(\mathbf{p}) = 1 \text{ for all } \mathbf{p} \text{ in the unit circle}$$

so that $\mathbf{w}(\mathbf{p}) = f(\mathbf{p})\mathbf{v}(\mathbf{p})$ is conservative? (4p)

Solution. (a) We calculate the integral using the definition. A parametrization of the unit circle is given by

$$\gamma(t) = \langle \cos(t), \sin(t) \rangle \ 0 \leq t \leq 2\pi.$$

Thus, the integral is given by

$$\int_0^{2\pi} \mathbf{v}(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t = \int_0^{2\pi} \langle -\sin(t), \cos(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \ \mathrm{d}t = \int_0^{2\pi} \sin^2(t) + \cos^2(t) dt = 2\pi.$$

(b) No. Denote the unit circle with C. Towards a contradiction, assume that such a function exists. Then we would have

$$\int_C \mathbf{w} \cdot d\mathbf{r} = 0,$$

since C is a closed curve and $\mathbf{w}$ by assumption is conservative. However, since f is equal to 1 everywhere on the unit circle, we have

$$\int_C \mathbf{w} \cdot d\mathbf{r} = \int_C f\mathbf{v} \cdot d\mathbf{r} = \int_C \mathbf{v} \cdot d\mathbf{r} \neq 0,$$

by (a). This is a contradiction, and thus, such a function cannot exist.

□