Problem 1

Find the limit, if it exists, or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+3xy+y^2}{x^2+y^2}$$

Solution

Approaching $(0,0)$ along the curve $x=0$, we obtain:

$$\frac{x^2+3xy+y^2}{x^2+y^2} = \frac{y^2}{y^2} = 1.$$

Approaching $(0,0)$ along the curve $x=y$, we obtain:

$$\frac{x^2+3xy+y^2}{x^2+y^2} = \frac{x^2+3x^2+x^2}{x^2+x^2} = \frac{5x^2}{2x^2} = \frac{5}{2} \neq 1$$

$\Longrightarrow$ $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+3xy+y^2}{x^2+y^2}$ does not exist.

$\square$

Problem 2.

Let $f(x,y) = xy$.

a) Find and classify critical points of f.

b) Find the absolute max and min values of on the region $D = \{(x,y) \mid 9x^2 + y^2 \le 18\}$.

Solution

a) $\nabla f = \langle y, x \rangle = 0$

$y = 0,\ x = 0.$

$(0,0)$ is the only critical point.

$D(0,0) = (f_{xx} f_{yy} - (f_{xy})^2)(0,0) = (0 \cdot 0 - 1^2)(0,0) = -1.$

$\Rightarrow$ $(0,0)$ is a saddle point.

b) Absolute max and min values must be either at critical points lying inside D or on the boundar

To find max and min values at the boundary of D, which is the circle $9x^2 + y^2 = 18$, we use Lagrange Multipliers Method.

Let $g(x,y) = 9x^2+y^2-18$

$$\nabla g = \langle 18x, 2y \rangle$$

The points of max and min on the boundary of D should satisfy the system of equations:

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y)=0 \end{cases} \Longrightarrow \begin{cases} \langle y, x\rangle = \lambda \langle 18x, 2y\rangle \\ 9x^2+y^2=18 \end{cases} \Longrightarrow$$

$$\begin{cases} y = 18\lambda x \\ x = 2\lambda y \\ 9x^2+y^2=18 \end{cases} \Longrightarrow \begin{cases} y = 18\lambda x \\ x = 36\lambda^2 x \\ 9x^2+y^2=18 \end{cases} \Longrightarrow$$

case 1: $x=0$, $y=18\lambda x=0$, but then the 3rd equation doesn't hold. So this case is impossible.

case 2: $36\lambda^2=1$
$\lambda = \pm\frac{1}{6}$ $\Longrightarrow \begin{cases} y = \pm 3x \\ 9x^2+9x^2=18 \end{cases} \Longrightarrow \begin{cases} y=\pm 3x \\ x^2=1 \end{cases}$

$\Longrightarrow x=\pm 1,\ y=\pm 3x \Longrightarrow$ we obtain 4 points:

$(1,3)$, $(1,-3)$, $(-1,3)$, $(-1,-3)$.

We compute values of f at these 4 points and also at the critical point $(0,0)$ (it lies inside D).

$f(0,0)=0$

$f(1,3)=3$

$f(-1,-3)=3$ $\implies$ the absolute max value is 3,

$f(1,-3)=-3$ the absolute min value is -3.

$f(-1,3)=-3$

$\square$

Problem 3

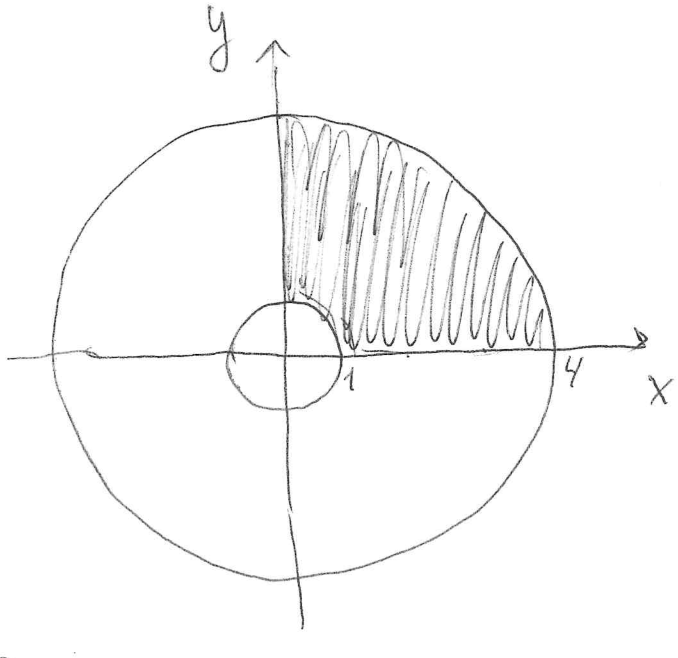
Compute the double integral

$$\iint_D y\,(x^2+y^2)^{3/2}\,dA,$$

where D is the region that lies in the first quadrant (i.e. $x, y \geq 0$) between the circles $x^2+y^2=1$ and $x^2+y^2=16$.

Solution

We will use polar coordinates.



$$\iint_D y\,(x^2+y^2)^{3/2}\,dA$$

$$= \int_0^{\frac{\pi}{2}} \int_1^4 r\sin\theta\,(r^2)^{3/2}\,r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_1^4 \sin\theta\, r^5\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{r^6}{6}\right]_{r=1}^{r=4} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin\theta\left(\frac{4^6}{6} - \frac{1}{6}\right) d\theta = \frac{4^6-1}{6}\,\left[-\cos\theta\right]_{\theta=0}^{\theta=\frac{\pi}{2}} = \frac{4^6-1}{6}.$$

$\square$.

Problem 4

Compute the volume of the solid

$$E = \{(x, y, z) \mid 0 \le y \le 1,\ y^2 \le x \le 1,\ -1 \le z \le xy\}$$

Solution

$$\text{volume of } E = \iiint_E dV = \int_0^1 \int_{y^2}^1 \int_{-1}^{xy} dz\, dx\, dy$$

$$= \int_0^1 \int_{y^2}^1 (xy+1)\, dx\, dy = \int_0^1 \left[\frac{x^2 y}{2} + x \right]_{x=y^2}^{x=1} dy$$

$$= \int_0^1 \left(\frac{y}{2} + 1 - \frac{y^5}{2} - y^2 \right) dy = \left[\frac{y^2}{4} + y - \frac{y^6}{12} - \frac{y^3}{3} \right]_{y=0}^{y=1}$$

$$= \frac{1}{4} + 1 - \frac{1}{12} - \frac{1}{3} = \frac{5}{6}.$$

□

Problem 5

Let C be the curve given by parametrization
$$\bar{r}(t) = 3\sin t\ \bar{i} + 3\cos t\ \bar{j} + 4t\ \bar{k}, \quad 0 \le t \le \pi.$$

a) Find the length of C

b) Find the work done by the force field
$$\bar{F} = -y\bar{i} + x\bar{j} + z\bar{k}$$
in moving an object along C.

Solution a) $\bar{r}(t) = \langle 3\sin t, 3\cos t, 4t \rangle$

$\bar{r}'(t) = \langle 3\cos t, -3\sin t, 4 \rangle$

$|\bar{r}'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{9+16} = 5$

$$L = \int_0^{\pi} |\bar{r}'(t)|\, dt = \int_0^{\pi} 5\, dt = 5\pi.$$

b) work $= \int_C \bar{F} \cdot d\bar{r} = \int_C -y\, dx + x\, dy + z\, dz$

$$= \int_0^{\pi} -y(t)\, x'(t)\, dt + x(t)\, y'(t)\, dt + z(t)\, z'(t)\, dt$$

$$= \int_0^{\pi} (-3\cos t)\, 3\cos t\, dt + 3\sin t\,(-3\sin t)\, dt + 4t \cdot 4\, dt$$

$$= \int_0^{\pi} (9\cos^2 t + 9\sin^2 t + 16t)\, dt = \int_0^{\pi} (9 + 16t)\, dt = \left[8t^2 - 9t\right]_{t=0}^{t=\pi} = 8\pi^2 - 9\pi$$

□

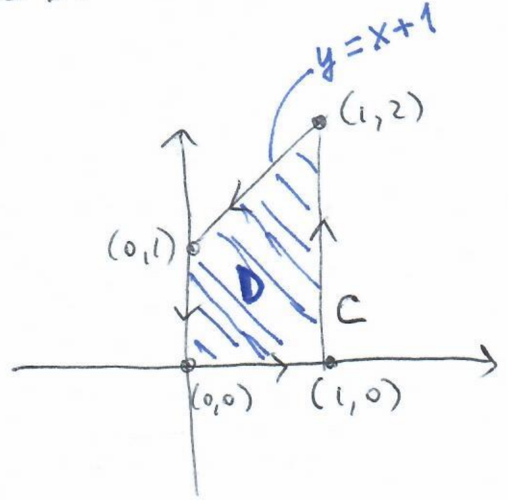
Problem 6

Compute the line integral

$$\int_C (x^2 y + e^{x^2})\, dx + (y^3 - \sin(e^y))\, dy,$$

where C is the trapezoidal curve consisting of the line segment going from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(1,2)$, from $(1,2)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Solution



Since C is a positively oriented closed curve, we can use Green's Theorem.

$$\int_C \underbrace{(x^2 y + e^{x^2})}_{P}\, dx + \underbrace{(y^3 - \sin(e^y))}_{Q}\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (-x^2)\, dA$$

$$= -\int_0^1 \int_0^{x+1} x^2\, dy\, dx = -\int_0^1 \left[x^2 y\right]_{y=0}^{x+1} dx = -\int_0^1 x^2(x+1)\, dx$$

$$= -\int_0^1 (x^3 + x^2)\, dx = -\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{x=0}^{x=1} = -\frac{1}{4} - \frac{1}{3} = -\frac{7}{12}.$$

$\square$.

Problem 7

Let $E = \{ (x,y,z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 2,\ 0 \leq z \leq 3 \}$

and let S be its surface given with the positive orientation. Let

$$\overline{F} = xy\,\overline{i} + xe^{z}\,\overline{j} + xe^{y}\,\overline{k}.$$

Find the flux of $\overline{F}$ across S, that is find

$$\iint_S \overline{F} \cdot d\overline{S}.$$

Solution $\quad \operatorname{div} \overline{F} = y.$

$$\iint_S \overline{F} \cdot d\overline{S} \overset{\text{Divergence Th.}}{=\!=\!=} \iiint_E \operatorname{div} \overline{F}\, dV$$

$$= \iiint_E y\, dV = \int_0^3 \int_0^2 \int_0^1 y\, dx\, dy\, dz = \int_0^3 \int_0^2 y\, dy\, dz$$

$$= \int_0^3 \left[\frac{y^2}{2} \right]_{y=0}^{y=2} dz = \int_0^3 2\, dz = 6. \qquad \square$$