Exam - Quantum optics and quantum information FKA173

Date and time: Tuesday, 27 October 2020, 8:30 - 12:30 + 30min for scanning the solutions
Zoom link for proctoring: https://chalmers.zoom.us/j/66883425079 you have to be on zoom with a webcam during the entire duration of the exam.
Examiners:

Witlef Wieczorek (witlef.wieczorek@chalmers.se)
Thilo Bauch (thilo.bauch@chalmers.se)
Giulia Ferrini (ferrini@chalmers.se)

Grading: There are 9 questions worth a total of 40 points. The points awarded for each question is indicated by that question and in the case of subquestions the division of the points is also indicated.
How to submit your solutions: You are expected to write down your solutions. All solutions must be uploaded to Canvas. In case you have written the solutions on paper, you must scan or photograph these written notes and upload them to Canvas. Please upload **all** your solutions as **one** document to Canvas.
General rules: all aids are allowed during the online remote exam. For example, you are allowed to access the internet and use any course material. However, this does not mean that you may copy text or figures from those sources. You are not allowed to plagiarize, but you are allowed to use the information you have found. **Write your online exam answers yourself, do not collaborate with another student or plagiarize! If you use information that you found on the internet or in the course literature (that is, permitted aids) do refer to the sources in your answer.**

Exam questions:

1. **Bloch sphere manipulations (3 pts.)**

   (a) (1 pt.) Write down a general qubit state using the angles θ and ϕ and depict it on the Bloch sphere.

   (b) (2 pt.) Given two pulsed d.c. fields $B_y(t)$ and $B_z(t)$ that couple to a qubit, i.e. the Hamiltonian is given by

$$\hat{\mathcal{H}} = -\frac{\hbar\gamma}{2}\left(B_y(t)\hat{\sigma}_y + B_z(t)\hat{\sigma}_z\right), \tag{1}$$

   with $\gamma > 0$, describe how to take the qubit from the state $|1\rangle$ to the state $\left(|0\rangle + i\,|1\rangle\right)/\sqrt{2}$. During how long time should the fields be applied?
   Draw the two states $|1\rangle$, $\left(|0\rangle + i\,|1\rangle\right)/\sqrt{2}$, the two magnetic fields, and the trajectories between the two states during operation on the Bloch sphere.
   Note: You can use the "left hand rule". Keep in mind that you can change the fields along y and z direction independently.

2. **Quantizing electrical circuits (7 pts.)**

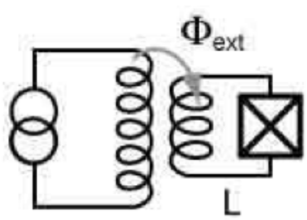

Figure 1: Circuit diagram of the RF SQUID

Derive the quantum mechanical Hamiltonian of the RF SQUID (see Fig. 1). In an RF SQUID the two sides of a superconducting tunnel junction with capacitance C_J are connected by a superconducting loop with inductance L. An external constant flux Φ_{ext} is imposed through the loop by an auxiliary coil. Here the tunnel junction is a parallel combination of a capacitance C_J and a Josephson junction with flux dependent potential energy $E_{JJ} = -E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$, where E_J is a constant and Φ the flux across the junction and $\Phi_0 = h/(2e)$ the superconducting flux quantum.

(a) (3 pts.) Write down the classical Lagrangian $\mathcal{L}(\Phi, \dot{\Phi})$ for the circuit. Use the flux through the inductance L as the independent variable.

(b) (1 pt.) Find the conjugate momentum to the flux coordinate. What is the physical meaning of the conjugate momentum?

(c) (2 pt.) From the Lagrangian derive the classical Hamiltonian of the circuit using the Legendre transformation.

(d) (1 pt.) Promoting the flux coordinate and its conjugate momentum to quantum mechanical operators one obtains the quantum mechanical Hamiltonian. Write down the commutation relation between the operators in the Hamiltonian.

3. **Phase space description of states of the light field (2.5 pts.)**

The quadrature operators $\hat{X}_1$ and $\hat{X}_2$ can be used to characterize states of the light field. They are defined as

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \tag{2}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger). \tag{3}$$

(a) (1 pt.) Consider a coherent state $|\alpha\rangle$ with $\alpha = |\alpha|e^{i\phi}$. Draw the state $|\alpha\rangle$ in phase space (i.e. $\langle\hat{X}_2\rangle$ vs. $\langle\hat{X}_1\rangle$) and label all parameters of the coherent state in the diagram including its uncertainties for $\hat{X}_1$ and $\hat{X}_2$. Draw this coherent state also after time t_1, i.e., $\phi \to \phi + \omega t_1$.

(b) (1 pt.) Draw qualitatively in real space a coherent state and an amplitude squeezed state. Real space is spanned by $\langle\hat{E}(t)\rangle$ vs. time with $\langle\hat{E}(t)\rangle$ the expectation value of the electric field operator. Indicate the uncertainty of the respective state as a shaded region around the mean value of $\hat{E}(t)$.

(c) (0.5 pts.) Name one possible application, where a squeezed state is superior to a coherent state. Explain, why the application you have chosen profits from using a squeezed state compared to using a coherent state.

4. **Measurement of non-classical light (2 pts.)**

(a) (2 pts.) Assume that you have two types of light sources. The first one emits photon packets that contain three photons at a time and the second one emits a coherent state. Explicitly derive the measurement result when you measure the normalized $g^{(1)}$ and $g^{(2)}$-functions of these two light sources.

Remember that

$$g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{\sqrt{G^{(1)}(x_1, x_1) \cdot G^{(1)}(x_2, x_2)}}, \tag{4}$$

$$g^{(2)}(x_1, x_2) = \frac{G^{(2)}(x_1, x_2)}{G^{(1)}(x_1, x_1) \cdot G^{(1)}(x_2, x_2)} \tag{5}$$

with the corresponding correlation functions given as

$$G^{(1)}(x_1, x_2) = \langle i|\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)|i\rangle \tag{6}$$
$$G^{(2)}(x_1, x_2) = \langle i|\hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)|i\rangle \tag{7}$$

and the electric fields $\hat{E}^{(+)}(x) = E_0\hat{a}e^{i(kx-\omega t)}$, $\hat{E}^{(-)}(x) = E_0\hat{a}^\dagger e^{-i(kx-\omega t)}$ and the initial state of the field $|i\rangle$.

5. **Rabi Hamiltonian (3.5 pts.)**

The Rabi Hamiltonian describes the interaction between light and a two-level system semi-classically and is given as

$$\hat{\mathcal{H}}_{\text{Rabi}} = -\frac{\hbar\omega_0}{2}\hat{\sigma}_z + \hbar\Omega_1 \cos(\omega t + \phi)\hat{\sigma}_x, \tag{8}$$

with the Pauli matrices $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) (1 pt.) Name one approximation that has been made when deriving the Rabi Hamiltonian. Explain its validity and explain under which circumstances this approximation breaks down.

(b) (2.5 pts.) The general solution of the Rabi problem can be written as

$$|\psi(t)\rangle = c_g(t)e^{-i\frac{E_g t}{\hbar}}|g\rangle + c_e(t)e^{-i\frac{E_e t}{\hbar}}|e\rangle \tag{9}$$

with

$$c_e(t) = c_+(t_0)e^{i\lambda_+ t} + c_-(t_0)e^{i\lambda_- t} \tag{10}$$
$$c_g(t) = -\frac{2}{i\Omega_1}e^{i\phi}e^{-i\Delta t}\left[i\lambda_+ c_+(t_0)e^{i\lambda_+ t} + i\lambda_- c_-(t_0)e^{i\lambda_- t}\right] \tag{11}$$

and

$$\lambda_\pm = \tfrac{1}{2}\left(\Delta \pm \underbrace{\sqrt{\Delta^2 + \Omega_1^2}}_{=:\Omega}\right) = \tfrac{1}{2}(\Delta \pm \Omega) \tag{12}$$

Assume that the atom is at time t_0 in the ground state, that the detuning is $\Delta = 0$ and the light phase is $\phi = 0$.

i. Explicitly calculate the atomic inversion $w(t) = P_e(t) - P_g(t)$ at any given time, with $P_i(t) = |\langle i|\psi(t)\rangle|^2$ and $i \in \{g, e\}$.

ii. At which times t is (i) the atom in the excited state, (ii) the atom in the ground state and (iii) when is $w(t) = 0$ (explicitly write down the times in multiples of π/Ω_1)? How would one call each of these pulses?

6. **Jaynes-Cummings Hamiltonian (5 pts.)**

The Jaynes-Cummings Hamiltonian can be derived from the Rabi Hamiltonian by replacing the classical field $E(t)$ of a single mode in a one dimensional cavity by the electric field operator $\hat{E}(t) = \hat{E}_x(z, t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz)$ resulting in

$$\hat{\mathcal{H}}_{\text{JC}} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-), \tag{13}$$

with the atomic raising and lowering operators $\hat{\sigma}_+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\hat{\sigma}_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, respectively, the Pauli operators $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and the atom-field coupling g.

(a) (1 pt.) Explicitly derive the Hamiltonian $\hat{\mathcal{H}}_{\text{JC}}^{(n-1)}$ in the subspace of the states $|g\rangle|n\rangle$ and $|e\rangle|n-1\rangle$ with $n > 0$.

(b) (0.5 pts.) The time evolution operator of Hamiltonian $\hat{\mathcal{H}}_{\text{JC}}^{(n)}$ in the subspace of $|g\rangle|n+1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e\rangle|n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is given by

$$\hat{U}(t) = \begin{pmatrix} \cos\left(\frac{\Omega_n}{2}t\right) + i \cdot n_z \sin\left(\frac{\Omega_n}{2}t\right) & i \cdot n_x \sin\left(\frac{\Omega_n}{2}t\right) \\ i \cdot n_x \sin\left(\frac{\Omega_n}{2}t\right) & \cos\left(\frac{\Omega_n}{2}t\right) - i \cdot n_z \sin\left(\frac{\Omega_n}{2}t\right) \end{pmatrix}, \tag{14}$$

with $\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}$, $n_z = \Delta/\Omega_n$ and $n_x = \frac{-2g\sqrt{n+1}}{\Omega_n}$ and the detuning $\Delta = \omega_0 - \omega$.

Let us assume that the initial state of the system is $|\psi(0)\rangle = |e\rangle|0\rangle$. What is the state after some time t?

(c) (2 pt.) What is the probability to find the state $|\psi(t)\rangle$ in the excited state $|e\rangle|0\rangle$? Draw the excited state probability versus time for $t \in \{0, \frac{\pi}{g}\}$ in case the detuning is $\Delta = 0$. Describe qualitatively what is happening in the first time interval $\{0, \frac{\pi}{2g}\}$ and in the second time interval $\{\frac{\pi}{2g}, \frac{\pi}{g}\}$.

(d) (1.5 pts.) The eigenergies $E^{\pm}(n)$ of $\hat{\mathcal{H}}_{\text{JC}}^{(n)}$ in the basis $|g\rangle|n+1\rangle$ and $|e\rangle|n\rangle$ are

$$E^{\pm}(n) = \hbar\omega\left(n + \frac{1}{2}\right) \pm \frac{\hbar}{2}\sqrt{\Delta^2 + 4g^2(n+1)}. \tag{15}$$

Draw a figure where you show the energies $E^+(0)$ and $E^-(0)$ *qualitatively* in dependence of the detuning $\Delta = \omega_0 - \omega$ (assume that only the atomic frequency ω_0 changes and that the frequency of the light field ω remains constant) for the coupled case ($g \neq 0$) and for the uncoupled case ($g = 0$). What is the major difference between the coupled light-atom system and the uncoupled light-atom system?

7. **Density matrix (2.5 pts.)**

The density matrix ρ incorporates mixed states into the formalism of quantum mechanics and allows, amongst others, to distinguish a pure from a mixed state. Consider the situation that Alice (A) and Bob (B) share a quantum state ρ_{AB}.

(a) (0.5 pts.) Assume that Alice and Bob share the state

$$\rho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|). \tag{16}$$

Do they share a pure or a mixed state? Why?

(b) (1 pt.) Now assume that Alice cannot talk to Bob anymore. What is the state of Alice, if she does not know about Bob's part of the state? Has Alice then a pure or a mixed state?

(c) (1 pt.) In case of a qubit, the density matrix can be visualized as a Bloch vector on the Bloch sphere with components $\vec{r} = (r_x, r_y, r_z)^T$ with $r_i = \langle\hat{\sigma}_i\rangle$. Which of the following states are pure or mixed **and** depict them on the Bloch sphere:

(i) $\vec{r}_1 = \left(\frac{1}{2}, 0, \frac{1}{2}\right)^T$.

(ii) $\vec{r}_2 = \left(\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{3}}, 0\right)^T$.

8. **Bloch equations (4.5 pts.)**

The Bloch equations describe the dynamics of a qubit under the presence of decoherence and can be written as

$$\frac{d\vec{r}}{dt} = \vec{r} \times \vec{\omega}(t) - \frac{r_z - r_z^0}{T_1}\hat{z} - \frac{r_x\hat{x} + r_y\hat{y}}{T_2}, \tag{17}$$

with the Hamiltonian of the qubit given as

$$\mathcal{H} = -\frac{\hbar}{2}\vec{\omega}(t)\vec{\sigma}, \tag{18}$$

with $\vec{\sigma} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)^T$, $\vec{\omega}(t) = (\omega_x(t), \omega_y(t), \omega_z(t))^T$ and $\hat{x}, \hat{y}, \hat{z}$ the unit vectors along the x, y, z direction of the Bloch sphere and $r_i = \langle\hat{\sigma}_i\rangle$ with $i \in \{x, y, z\}$. Assume that the qubit is quantized along the z direction.

(a) (1 pt.) What is the meaning of the time constants T_1 and T_2 in the Bloch equations? Which noise processes are relevant for the T_1 time and which for the T_2 time?

(b) (1.5 pts.) Assume that the qubit's Hamiltonian is given as $\mathcal{H} = -\frac{\hbar}{2}\omega_z\hat{\sigma}_z$ and that the qubit is initially (time $t = 0$) in its ground state $|g\rangle$. Further assume that the qubit's equilibrium population along the z direction is given as $r_z^0 = 1$ (i.e. the ground state $|g\rangle$) and that both time scales T_1 and T_2 are infinite.

Write down the Bloch equations for this special case and calculate the time-dependent atomic inversion, i.e., $w(t) = P_e(t) - P_g(t)$ with $P_e(t) = \frac{1}{2}(1 - r_z(t))$.

(c) (2 pts.) Now assume that T_1 and T_2 are finite and that the qubit is initially (time $t = 0$) in its excited state $|e\rangle$ (with the qubit Hamiltonian $\mathcal{H} = -\frac{\hbar}{2}\omega_z\hat{\sigma}_z$ and $r_z^0 = 1$). Write down the Bloch equations for this special case and calculate the time-dependent atomic inversion $w(t)$. You can use that the solution of $f'(x) = -f(x)/a + 1/a$ is $f(x) = 1 + e^{-x/a}(f(0) - 1)$. At which time is the atomic inversion equal to zero?

9. **Quantum algorithms (10 points)**

Consider the circuit represented in Fig. 2, where the unknown state $|\psi_s\rangle = \alpha|0\rangle + \beta|1\rangle$ is entangled with a multiqubit state $|C\rangle$. The state $|C\rangle$ is a 5-qubit state, that we will call the code state. The C_Z gate in the circuit represents a simultaneous C_Z operation acting on all qubits composing the code state. Let us introduce the following notation for two different logical states of the code:

$$|0_L\rangle \equiv |C\rangle \tag{19}$$

$$|1_L\rangle \equiv \hat{Z}|C\rangle, \tag{20}$$

where $\hat{Z} \equiv \hat{Z}_1\hat{Z}_2\hat{Z}_3\hat{Z}_4\hat{Z}_5$.

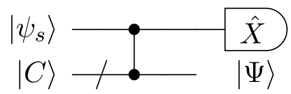

Figure 2: Circuit realising the encoding of an unknown state onto several physical qubits. The state $|C\rangle$ is a multiqubit state, here a 5 qubit state, as indicated by the multiqubit (crossed) quantum wire.

(a) (2 pts.) Compute the combined state $|\Psi\rangle_{\text{tot}}$ of single qubit and code state before the measurement.

(b) (2 pts.) Express this state $|\Psi\rangle_{\text{tot}}$ using the eigenbasis of the Pauli matrix $\hat{X}$, i.e., the states $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$, to express the state of the first qubit, and the logical states in Eq. (19) for the code state.

(c) (2 pts.) The observable $\hat{X}$ is then measured on the first qubit. Compute the state of the multiqubit system after the measurement, depending on whether a measurement outcome 0 or 1 is obtained when the measurement is performed, and express it in terms of the two multiqubit states $|0_L\rangle$ and $|1_L\rangle$.

Suppose that the outcome 1 is obtained. The unknown state has then been encoded successfully onto the code state. We now design a subset of 3 qubits among the qubits composing the code, namely we will consider the 3 first qubits, labeled by 1,2,3. We want to show that if these 3 qubits are individually measured in a suitable way, we can obtain the same information as if the unknown state $|\psi_s\rangle$ had been measured. In other words, we want to find logical operators $\hat{X}_L^{1,2,3}$ and $\hat{Z}_L^{1,2,3}$ that act on the code state as regular Pauli operators, i.e.

$$\begin{aligned} \hat{X}_L^{1,2,3}|0_L\rangle = |1_L\rangle; \;\; \hat{X}_L^{1,2,3}|1_L\rangle = |0_L\rangle \\ \hat{Z}_L^{1,2,3}|0_L\rangle = |0_L\rangle; \;\; Z_L^{1,2,3}|1_L\rangle = (-1)|1_L\rangle. \end{aligned} \tag{21}$$

(d) (2 pts.) Using the definitions that we gave above, show that the following operator accomplishes the desired operations in the first line of Eq.(21):

$$\hat{X}_L^{1,2,3} = (\hat{Z}_1\hat{X}_1)(\hat{Z}_2)(\hat{Z}_3\hat{X}_3). \tag{22}$$

Hint: assume and use that, for this special multiqubit state, $|0_L\rangle = \hat{K}_1\hat{K}_3|0_L\rangle$ with $\hat{K}_1 = \hat{X}_1\hat{Z}_5\hat{Z}_2$ and $\hat{K}_3 = \hat{X}_3\hat{Z}_4\hat{Z}_2$, as well as the properties of Pauli matrices such as $\hat{Z}_i^2 = \hat{X}_i^2 = 1$, $\hat{X}_i\hat{Z}_j = (-1)^{\delta_{i,j}}\hat{Z}_j\hat{X}_i$. For completeness, we also report that $\hat{Z}_L^{1,2,3} = \hat{X}_2\hat{Z}_1\hat{Z}_3$ (we won't be interested in this demonstration here, because it requires to know the structure of the multiqubit state $|C\rangle$).

(e) (2 pts.) In the above questions we have seen an example of a protocol where information from a subset of k qubits (3 in our case) out of an $n-$qubit code state (composed of 5 qubits in our case) allows for reconstructing the unknown state. In general, this reconstruction is possible when $n < 2k$, where k is the number of qubits in the subset, and n is the total number of qubits in the code state. Can you explain by using theorems that you know why it must be impossible to have a $n = 2k$ protocol instead?