Quantum Optics and Quantum Informatics FKA173

Date and time: Tuesday, 29 October 2019, 8:30 - 12:30
Examiners: Giulia Ferrini (0709492347), Witlef Wieczorek (0733-873214), and Thilo Bauch (0733-66 13 79). Visits around 09:30 and 11:30
Grading: There are 8 questions worth a total of 40 points. The points awarded for each question is indicated by that question and in the case of subquestions the division of the points is also indicated.

1. **Bloch sphere manipulations (3 pts.)**

   (a) (1 pt.) Write down a general qubit state using the angles θ and ϕ and depict it on the Bloch sphere.

   (b) (2 pt.) Given two pulsed d.c. fields $B_y(t)$ and $B_z(t)$ that couple to a qubit, i.e. the Hamiltonian is given by

   $$\hat{H} = -\frac{\hbar\gamma}{2}\left(B_y(t)\widehat{\sigma}_y + B_z(t)\widehat{\sigma}_z\right),$$

   with $\gamma > 0$, describe how to take the qubit from the state $|0\rangle$ to the state $\left(|0\rangle - i\,|1\rangle\right)/\sqrt{2}$. During how long time should the fields be applied?
   Note: You can use the "left hand rule" together with the time-dependent rotation angle $\delta = \gamma\int_0^t dt'|\mathbf{B}|(t')$ around the field axis. Keep in mind that you can change the fields along y and z direction independently.

2. **Quantizing electrical circuits (7 pts.)**

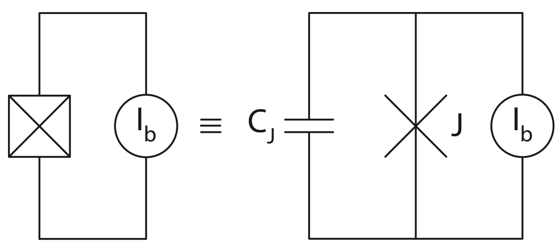

Figure 1: Circuit diagram of the current biased Josephson junction

Derive the quantum mechanical Hamiltonian of the current biased Josephson junction (see Fig. 1). The potential energy of the Josephson junction, represented by J, is given by $E_{JJ} = -E_J\cos\left(2\pi\frac{\Phi}{\Phi_0}\right)$, where E_J is a constant and Φ the flux across the junction and $\Phi_0 = h/(2e)$ the superconducting flux quantum. The capacitance of the junction is given by C_J. I_b represents the external constant current source. The bias current source can be considered as an inductor L, which is threaded by an external flux $\tilde{\Phi}$, and taking $L \to \infty$ and $\tilde{\Phi} \to \infty$, while keeping $\tilde{\Phi}/L = I_b$.

   (a) (3 pts.) Derive the classical Lagrangian $\mathcal{L}(\Phi, \dot{\Phi})$ for the circuit using the flux. Use the flux through the Josephson junction as the independent variable.

   (b) (1 pt.) Find the conjugate momentum to the flux coordinate. What is the physical meaning of the conjugate momentum?
   Note: The conjugate momentum of the flux Φ is given by $q = \partial\mathcal{L}/\partial\dot{\Phi}$.

   (c) (2 pt.) Write down the classical Hamiltonian for the circuit.
   Note: Use the Legendre transformation to obtain the Hamiltonian: $H(\Phi_n, q_n) = \sum_n \dot{\Phi}_n q_n - \mathcal{L}$

   (d) (1 pt.) Promoting the flux coordinate and its conjugate momentum to quantum mechanical operators one obtains the quantum mechanical Hamiltonian. What are the commutation relations between the operators in the Hamiltonian?

3. **Phase space description of states of the light field (2 pts.)**

The quadrature operators $\hat{X}_1$ and $\hat{X}_2$ can be used to characterize states of the light field. They are defined as

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \tag{1}$$

$$\hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger). \tag{2}$$

(a) (1 pt.) Consider a coherent state $|\alpha\rangle$ with $\alpha = |\alpha|e^{i\phi}$. Draw the state $|\alpha\rangle$ in phase space (i.e. $\langle\hat{X}_2\rangle$ vs. $\langle\hat{X}_1\rangle$) including its uncertainties for $\hat{X}_1$ and $\hat{X}_2$.

(b) (1 pt.) Draw qualitatively in phase space an amplitude squeezed vacuum state including its uncertainties for $\hat{X}_1$ and $\hat{X}_2$.

4. **Detection of non-classical light (3.5 pts.)**

(a) (2 pts.) Which experiment would you use to distinguish a single photon light source from a coherent state light source. Explain how this experiment works and why it allows you to distinguish these two states.

(b) (1.5 pts.) Assume that you have a photon light source that emits the Fock state $|2\rangle$. Derive the result when you measure the normalized $g^{(2)}$-function of the Fock state $|2\rangle$. Remember that

$$g^{(2)}(x_1, x_2) = \frac{G^{(2)}(x_1, x_2)}{G^{(1)}(x_1, x_1) \cdot G^{(1)}(x_2, x_2)} \tag{3}$$

with the corresponding correlation functions given as

$$G^{(1)}(x_1, x_2) = \langle i|\, \hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)\, |i\rangle \tag{4}$$

$$G^{(2)}(x_1, x_2) = \langle i|\, \hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)\, |i\rangle \tag{5}$$

and the electric fields $\hat{E}^{(+)}(x) = E_0\hat{a}e^{i(kx-\omega t)}$, $\hat{E}^{(-)}(x) = E_0\hat{a}^\dagger e^{-i(kx-\omega t)}$ and the initial state of the field $|i\rangle$.

5. **Jaynes-Cummings Hamiltonian (9 pts.)**

The Jaynes-Cummings Hamiltonian can be derived from the Rabi Hamiltonian by replacing the classical field $\mathbf{E}(t)$ of a single mode in a one dimensional cavity by an operator $\hat{\mathbf{E}}(t) = \hat{E}_x(z,t) = \mathcal{E}_0(\hat{a}+\hat{a}^\dagger)\sin(kz)$, resulting in

$$\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-).$$

(a) (2 pts.) Derive the Hamiltonian $\hat{H}^{(n)}$ in the subspace of the states $|g\rangle|n+1\rangle$ and $|e\rangle|n\rangle$. Remember that $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$.

(b) (1.5 pts.) The time evolution operator of Hamiltonian $\hat{H}^{(n)}$ in the subspace of $|g\rangle|n+1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|e\rangle|n\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is given by

$$\hat{U}(t) = \begin{pmatrix} \cos\left(\frac{\Omega_n}{2}t\right) + i\cdot n_z \sin\left(\frac{\Omega_n}{2}t\right) & i\cdot n_x \sin\left(\frac{\Omega_n}{2}t\right) \\ i\cdot n_x \sin\left(\frac{\Omega_n}{2}t\right) & \cos\left(\frac{\Omega_n}{2}t\right) - i\cdot n_z \sin\left(\frac{\Omega_n}{2}t\right) \end{pmatrix}, \tag{6}$$

with $\Omega_n = 2g\sqrt{n+1}$, $n_z = \Delta/\Omega_n$ and $n_x = \frac{-2g\sqrt{n+1}}{\sqrt{\Delta^2+4g^2(n+1)}}$ and the detuning $\Delta = \omega_0 - \omega$. What is the state $|\psi(t)\rangle$ after some time t given the initial state $|\psi(0)\rangle = |e\rangle|n\rangle$ and for zero detuning $\Delta = 0$?

(c) (2.5 pts.) Consider now the case for $n = 0$. What is the probability to find the state $|\psi(t)\rangle$ in the excited state $|e\rangle|0\rangle$? Draw the excited state probability versus time for $t \in \{0, \frac{\pi}{g}\}$. Describe qualitatively what is happening in the first time interval $\{0, \frac{\pi}{2g}\}$ and in the second time interval $\{\frac{\pi}{2g}, \frac{\pi}{g}\}$.

(d) (3 pts.) The eigenergies $E^{\pm}$ of $\hat{H}^{(n)}$ are

$$E^{\pm} = \hbar\omega(n + \frac{1}{2}) \pm \frac{\hbar}{2}\sqrt{(\omega_0 - \omega)^2 + 4g^2(n+1)}. \tag{7}$$

Consider the case for zero detuning, i.e., $\omega = \omega_0$. Depict the energy spectrum of the uncoupled ($g = 0$) and coupled ($g \neq 0$) atom-photon states. To this end draw a diagram of the photon energy ladder on top of the ground state $|g\rangle$ and excited state $|e\rangle$. Put the relevant energy differences of the atom energy, photon energy, and the energy difference between E^+ and E^- for $n = 0$ and $n = 1$ into the drawing.

6. **Density matrix (2.5 pts.)**

The density matrix ρ incorporates mixed states into the formalism of quantum mechanics and allows, amongst others, to distinguish a pure from a mixed state. Consider the situation that Alice (A) and Bob (B) share a quantum state $\rho_{AB} = |\psi\rangle_{AB} \langle\psi|_{AB}$.

(a) (1 pt.) Assume that Alice and Bob share the state

$$\rho_{AB} = \frac{1}{2}(|00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11|). \tag{8}$$

Do they share a pure or a mixed state? Why?

(b) (1.5 pts.) Now assume that Alice cannot talk to Bob anymore. What is the state of Alice, if she does not know about Bob's part of the state? Has Alice then a pure or a mixed state?

7. **Bloch equations (3 pts.)**

The Bloch equations describe the dynamics of a qubit under the presence of decoherence and can be written as

$$\frac{d\vec{r}}{dt} = \vec{r} \times \vec{\omega}(t) - \frac{r_z - r_z^0}{T_1}\hat{z} - \frac{r_x\hat{x} + r_y\hat{y}}{T_2},$$

with the Hamiltonian of the qubit given as

$$\mathcal{H} = -\frac{\hbar}{2}\vec{\omega}(t)\vec{\sigma},$$

with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$, $\vec{\omega}(t) = (\omega_x(t), \omega_y(t), \omega_z(t))^T$ and $\hat{x}, \hat{y}, \hat{z}$ the unit vectors along the x, y, z direction of the Bloch sphere and $r_i = \langle\sigma_i\rangle$ with $i \in \{x, y, z\}$.

(a) (1.5 pts.) Assume that the qubit's Hamiltonian is given as $\mathcal{H} = -\frac{\hbar}{2}\omega_z\sigma_z$ and that the qubit is initially (time $t = 0$) in its excited state $|e\rangle$. Further assume that the qubit's equilibrium population along the $z-$direction is given as $r_z^0 = 1$ (i.e. the ground state $|g\rangle$) and that both time scales T_1 and T_2 are infinite.

Write down the Bloch equations for this special case and calculate the time-dependent occupation of the excited state, i.e., $P_e(t) = \frac{1}{2}(1 - r_z(t))$.

Note: Remember that the vector product between two vectors is defined as $\vec{a}\times\vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)^T$.

(b) (1.5 pts.) Now assume that T_1 and T_2 are finite. Write down the Bloch equations for this special case and calculate the time-dependent occupation of the excited state. You can use that the solution of $f'(x) = -f(x)/a + 1/a$ is $f(x) = 1 + e^{-x/a}(f(0) - 1)$.

8. **Optimal cloning machine (10 pts.)**

(a) (2 pts.) Recall the statement and the proof of the no-cloning theorem. Does this theorem say anything about the possibility of realizing imperfect copies of a quantum states?

(b) (2 pts.) Let's consider a machine acting on the three qubits in the following way:

$$|000\rangle_{ABC} \to |\chi_0\rangle = \sqrt{\frac{2}{3}}\,|00\rangle_{AB}\,|0\rangle_C + \sqrt{\frac{1}{3}}\,|\beta_{01}\rangle_{AB}\,|1\rangle_C$$

$$|100\rangle_{ABC} \to |\chi_1\rangle = \sqrt{\frac{2}{3}}\,|11\rangle_{AB}\,|1\rangle_C + \sqrt{\frac{1}{3}}\,|\beta_{01}\rangle_{AB}\,|0\rangle_C$$

where $|\beta_{01}\rangle_{AB}$ is the Bell state:

$$|\beta_{01}\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}).$$

It is clear according to the notations that the qubit A is copied on the qubit B, the qubit C being an auxiliary for calculation.

Check that the scalar products $\langle 000|\,000\rangle$, $\langle 100|\,100\rangle$ and $\langle 000|\,100\rangle$ are conserved in this operation. Is this transformation physically realizable ?

(c) (2 pts.) What is the transformed state $|\psi^t\rangle_{ABC}$ obtained for an initial arbitraty state $|\Psi\rangle_A = a\,|0\rangle_A + b\,|1\rangle_A$ of the qubit A ? What is the density matrix of the initial state $|\Psi\rangle_A$?

(d) (2 pts.) To implement this optimal cloning machine, we consider the following circuit on the figure hereunder (see Fig. 2), where the usual notation for C-NOT gates is used. We will admit that the preparation part (with 3 rotations and 2 C-NOT) prepares qubits B and C in the state:

$$|\Psi\rangle_{BC} = \frac{1}{\sqrt{6}}(2\,|00\rangle + |01\rangle + |11\rangle).$$

Show that the copying circuit realizes the desired operation.

(e) (2 pts.) The reduced transformed density matrix for qubit B is defined as

$$\rho^t_B = \mathrm{Tr}_{AC}[\rho^t_{ABC}] = {}_{AC}\langle 00|\rho^t_{ABC}|00\rangle_{AC} + {}_{AC}\langle 01|\rho^t_{ABC}|01\rangle_{AC} + {}_{AC}\langle 10|\rho^t_{ABC}|10\rangle_{AC} + {}_{AC}\langle 11|\rho^t_{ABC}|11\rangle_{AC},$$

and analogously for $\rho^t_A = \mathrm{Tr}_{BC}[\rho^t_{ABC}]$. Given the transformed state $|\psi^t\rangle_{ABC}$ computed in point 3, compute the explicit expression of the reduced density matrix of qubit B. Show that it realizes an approximated copy of the initial qubit A, by comparing the two density matrices.

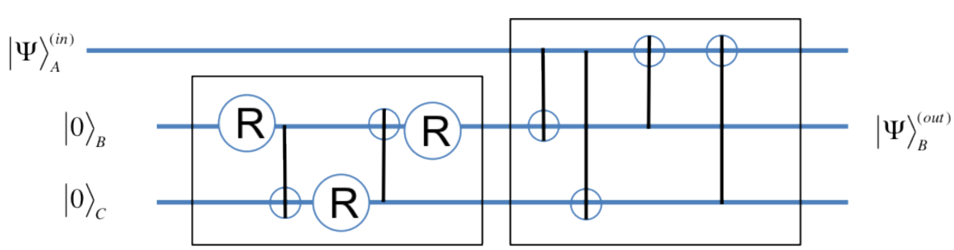

Figure 2: Optimal cloning machine