Quantum Optics and Quantum Informatics FKA173

Date and time: Tuesday, 30 October 2018, 08:30-12:30.
Examiners: Giulia Ferrini (0709492347), Witlef Wieczorek (0733-873214), and Thilo Bauch (0733-66 13 79). Visits around 09:30 and 11:30.
Grading: There are 6 questions worth a total of 20 points. The points awarded for each question is indicated by that question and in the case of subquestions the division of the points is also indicated.

1. **Bloch sphere manipulations (1.5 pts.)**

   (a) (0.5 pt.) Write down a general qubit state using the angles θ and ϕ and depict it on the Bloch sphere.

   (b) (1 pt.) Given two pulsed d.c. fields $B_x(t)$ and $B_z(t)$ that couple to a qubit, i.e. the Hamiltonian is given by

   $$\hat{H} = -\frac{\hbar\gamma}{2}\left(B_x(t)\widehat{\sigma}_x + B_z(t)\widehat{\sigma}_z\right),$$

   with $\gamma > 0$, describe how to take the qubit from the state $|0\rangle$ to the state $\left(|0\rangle - |1\rangle\right)/\sqrt{2}$. During how long time should the fields be applied?
   Note: You can use the "left hand rule" together with the time-dependent rotation angle $\delta = \gamma\int_0^t dt'|\mathbf{B}|(t')$ around the field axis. Keep in mind that you can change the fields along x and z direction independently.

2. **Quantizing Electrical Circuits (3.5 pts.)**

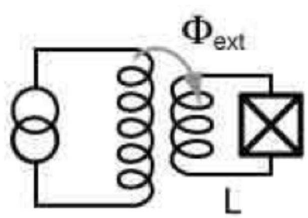

Figure 1: Circuit diagram of the RF SQUID

Derive the quantum mechanical Hamiltonian of the RF SQUID (see Fig. 1). In an RF SQUID the two sides of a superconducting tunnel junction with capacitance C_J are connected by a superconducting loop with inductance L. An external constant flux Φ_{ext} is imposed through the loop by an auxiliary coil. Here the tunnel junction is a parallel combination of a capacitance C_J and a Josephson junction with flux dependent potential energy $E_{JJ} = -E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$, where E_J is a constant and Φ the flux across the junction and $\Phi_0 = h/(2e)$ the superconducting flux quantum.

(a) (1.5 pts.) Write down the classical Lagrangian $\mathcal{L}(\Phi, \dot{\Phi})$ for the circuit. Use the flux throug the inductance L as the independent variable.

(b) (0.5 pt.) Find the conjugate momentum to the flux coordinate. What is the physical meaning of the conjugate momentum?
Note: The conjugate momentum of the flux Φ is given by $q = \partial\mathcal{L}/\partial\dot{\Phi}$.

(c) (1 pt.) Write down the classical Hamiltonian for the circuit.
Note: Use the Legendre transformation to obtain the Hamiltonian: $H(\Phi_n, q_n) = \sum_n \dot{\Phi}_n q_n - \mathcal{L}$

(d) (0.5 pt.) Promoting the flux coordinate and its conjugate momentum to quantum mechanical operators one obtains the quantum mechanical Hamiltonian. What are the commutation relations between the operators in the Hamiltonian?

3. **Coherent states (1 pts.)**

The coherent states $|\alpha(t)\rangle$ are the eigenstates of the annihilation operator

$$\hat{a}|\alpha(t)\rangle = \alpha(t)|\alpha(t)\rangle$$

with

$$|\alpha(t)\rangle = e^{-|\alpha(t)|^2/2} \sum_{n=0}^{\infty} \frac{\alpha(t)^n}{\sqrt{n!}} |n\rangle$$

and $\alpha(t) = e^{-i\omega t}\alpha$ and α is a complex number.

(a) (0.5 pt.) Calculate the expectation value

$$\langle \hat{E}_x(z,t)\rangle = \langle \alpha(t)|\hat{E}_x(z,t)|\alpha(t)\rangle,$$

i.e., the expectation value of the quantized electric field

$$\hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz)$$

for a coherent state $|\alpha(t)\rangle$.

(b) (0.5 pt.) What type of light source creates such coherent states? Name at least one other type of light state of the quantized radiation field (besides the coherent state).

4. **Jaynes-Cummings Hamiltonian (3.5 pts.)**

(a) The Jaynes-Cummings Hamiltonian can be derived from the Rabi Hamiltonian by replacing the classical field $\mathbf{E}(t)$ of a single mode in a one dimensional cavity (resonator) by an operator $\hat{\mathbf{E}}(t) = \hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz)$, resulting in

$$\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-).$$

(i) (0.5 pt.) What are the meanings of the three terms in the Hamiltonian?

(ii) (0.5 pt.) In the two-dimensional subspace spanned by $|g\rangle|n+1\rangle$ and $|e\rangle|n\rangle$, the Jaynes-Cummings Hamiltonian can be written as

$$\hat{H}_n = \hbar\omega\left(n + \frac{1}{2}\right)\hat{1} - \frac{\hbar(\omega_0 - \omega)}{2}\sigma_z + \hbar g\sqrt{n+1}\sigma_x,$$

where $|e\rangle$ and $|g\rangle$ are respectively the excited and ground state of the atom, and $|n\rangle$ is the photon number state. Here the Pauli operators and the unity matrix $\hat{1}$ are represented in the $|g\rangle|n+1\rangle$ and $|e\rangle|n\rangle$ basis and one can define the atom-field detuning as $\Delta = \omega_0 - \omega$.

Derive the Eigenenergies $E^{\pm}(\omega, \Delta, g)$ of the Hamiltonian as a function of ω, Δ, g.

Note: The eigenvalues $\lambda^{\pm}$ of a 2×2 matrix can be found from

$$\left(a_{11} - \lambda^{\pm}\right)\left(a_{22} - \lambda^{\pm}\right) - a_{12}a_{21} = 0,$$

where a_{11}, a_{22} are the diagonal and a_{12}, a_{21} the off-diagonal elements of the 2×2 matrix. The solution to a quadratic equation with $\lambda^2 + p\lambda + q = 0$ is given as $\lambda^{\pm} = -p/2 \pm \sqrt{(p/2)^2 - q}$.

(iii) (1 pt.) Having derived the eigenergies $E^{\pm}$, consider now the case for zero detuning, i.e., $\omega = \omega_0$. Depict the energy spectrum of the uncoupled ($g = 0$) and coupled ($g \neq 0$) atom-photon states. Put the relevant energy differences into the drawings.

(b) For large detuning between the field frequency and the atomic transition frequency ($\Delta = \omega_0 - \omega$, $|\Delta| \gg g$) we obtain the dispersive Hamiltonian:

$$\hat{H}_{\text{disp}} = -\frac{1}{2}\left(\hbar\omega_0 + \frac{\hbar g^2}{\Delta}\right)\sigma_z + \left(\hbar\omega - \frac{\hbar g^2}{\Delta}\sigma_z\right)\hat{a}^\dagger\hat{a},$$

where σ_z is represented in the atom basis.

(i) (0.5 pt.) What is the effective frequency of the atom when there are 5 photons in the cavity?

(ii) (1 pt.) The dispersive Hamiltonian can be used to read out the atom/qubit state using coherent (microwave) photons. Explain explicitly how such a measurement can be implemented and what is measured, and how the qubit state can be detected.

5. **Density matrix and the Bloch equations (3 pts.)**

(a) (0.5 pt.) The density matrix formalism incorporates mixed states into the formalism of quantum mechanics and allows, amongst others, to distinguish a pure from a mixed state.

Decide, which of the following states are pure or mixed:

(i) $\rho_1 = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$

(ii) $\rho_2 = \frac{1}{2}\left(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|\right)$

(b) (0.5 pt.) In case of a qubit, the density matrix can be visualized as a Bloch vector on the Bloch sphere with components $\vec{r} = (r_x, r_y, r_z)^T$ with $r_i = \langle \sigma_i \rangle$.

Decide, which of the following states are pure or mixed and depict them (approximately) on the Bloch sphere:

(i) $\vec{r}_3 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0)^T$.

(ii) $\vec{r}_4 = (0.7, 0.7, 0)^T$.

(c) The Bloch equations describe the dynamics of a qubit under the presence of decoherence and can be written as

$$\frac{d\vec{r}}{dt} = \vec{r} \times \vec{\omega}(t) - \frac{r_z - r_z^0}{T_1}\hat{z} - \frac{r_x\hat{x} + r_y\hat{y}}{T_2},$$

with the Hamiltonian of the qubit given as

$$\mathcal{H} = -\frac{\hbar}{2}\vec{\omega}(t)\vec{\sigma},$$

with $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)^T$, $\vec{\omega}(t) = (\omega_x(t), \omega_y(t), \omega_z(t))^T$ and $\hat{x}, \hat{y}, \hat{z}$ the unit vectors along the x, y, z direction of the Bloch sphere.

(i) (0.5 pt.) What is the meaning of the time constants T_1 and T_2 in the Bloch equations?

(ii) (0.5 pt.) Assume that the qubit's Hamiltonian is given as $\mathcal{H} = -\frac{\hbar}{2}\omega_z\sigma_z$ and that the qubit is initially (time $t = 0$) in its excited state $|e\rangle$. Further assume that the qubit's equilibrium population along the $z-$direction is given as $r_z^0 = 0$ and that both T_1 and T_2 are infinite.

Write down the Bloch equations for this special case and calculate the time-dependent occupation of the excited state, i.e., $P_e(t) = \frac{1}{2}(1 - r_z(t))$.

Note: Remember that the vector product between two vectors is defined as $\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)^T$.

(iii) (1 pt.) Now assume that T_1 and T_2 are finite. Write down the Bloch equations for this special case and calculate the time-dependent occupation of the excited state, i.e., $P_e(t) = \frac{1}{2}(1 - r_z(t))$. What is different to the previous solution and why?

6. **Measurement based Quantum Computation (7.5 pts.)**

Measurement based quantum computation (MBQC), also referred to as "one-way quantum computation", is an alternative but computationally equivalent model for quantum computation, with respect to the circuit model. In MBQC, applying unitary transformations to the input state

$|\psi_{\rm out}\rangle = U|\psi_{\rm in}\rangle$ is achieved by entangling it to a (large) resource state, said the "cluster state", and later local projective measurements according to suitable observables are performed onto the qubits of the cluster state. The observables to measure are chosen so to induce the desired transformation on the input state.

We recall the definition of the Pauli matrices $\hat{\sigma}_k$: renamed as $\hat{X}, \hat{Y}, \hat{Z}$, i.e.

$$\begin{aligned}
\hat{X} &= (|0\rangle\langle 1| + |1\rangle\langle 0|) \doteq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\hat{Y} &= (|0\rangle\langle 1| - |1\rangle\langle 0|) \doteq \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
\hat{Z} &= (|0\rangle\langle 0| - |1\rangle\langle 1|) \doteq \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned} \tag{1}$$

Rotations:

$$U_k(\phi) = e^{-\frac{i\phi}{2}\hat{\sigma}_k} \tag{2}$$

E.g. $U_z(\phi) = e^{-\frac{i\phi\hat{Z}}{2}}$.
Hadamard gate:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{3}$$

(a) **First steps: implementation of the identity**
We consider the general single-qubit state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. A second qubit is prepared in the state $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. A C_Z gate is applied to the two qubits, with $C_Z = |0\rangle\langle 0| \otimes 1 + |1\rangle\langle 1| \otimes \hat{Z}$.

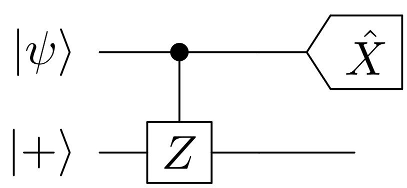

Figure 2: Scheme representing the elementary quantum circuit for teleportation. The box at the end of a line represents the measurement of the corresponding qubit according to the observable indicated in the box (in this case $\hat{X}$).

i. (2 pts.) Compute the resulting state expressing the second qubit in the $|\pm\rangle$ basis.

ii. (2 pts.) In order to implement the identity operator on the input qubit, after the C_Z gate the first qubit is measured in the $|\pm\rangle$ basis, i.e. we measure $\hat{X}$, see Fig.2. After the measurement, the second qubit state is projected onto a certain state. Compute in which state the second qubit is projected. As compared to the input state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, how has this output state being trasformed? Express the result in terms of the Hadamard gate and $\hat{X}^m$, where $m = 0$ if the measurement outcome obtained was 1, and $m = 1$ if -1.

These "extra" gates obtained on the input qubit are said "by-product" operators; they are undesired results of the teleportation procedure described above. However, they can be corrected for, or simply their effect can be taken into account by re-interpreting the measurement results at the end of the computation.

(b) **Implementation of a single qubit rotation**

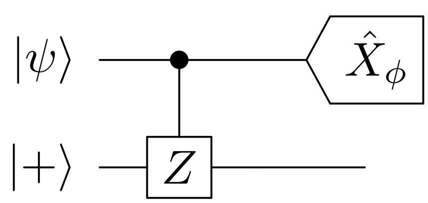

Figure 3: Scheme representing the elementary quantum circuit for *gate* teleportation. The box at the end of a line represents the measurement of the corresponding qubit according to the observable indicated in the box (in this case $U_z(\phi)\hat{X}U_z(-\phi)$).

i. (1 pt.) Now consider the observable $\hat{X}_\phi \equiv U_z^\dagger(\phi)\hat{X}U_z(\phi) = \cos\phi\hat{X} - \sin\phi Y = e^{i\phi}|0\rangle\langle 1| + e^{-i\phi}|1\rangle\langle 0|$. Verify that this operator can be equivalently written as $|\phi_+\rangle\langle\phi_+| - |\phi_-\rangle\langle\phi_-|$ where we have introduced the rotated basis $|\phi_\pm\rangle = 1/\sqrt{2}(|0\rangle \pm e^{-i\phi}|1\rangle)$.

ii. (2 pts.) This observable $U_z(-\phi)\hat{X}U_z(\phi) = |\phi_+\rangle\langle\phi_+| - |\phi_-\rangle\langle\phi_-|$ is measured in the first qubit after the C_Z gate, see Fig.3. Compute what is the state projected onto the second qubit. As done before, express the result in terms of the Hadamard gate and $\hat{X}^m$, where $m = 0$ if the measurement outcome obtained was 1, and $m = 1$ if -1, and of the rotation $U_z(\phi)$.

This procedure can be concatenated in a clever way to obtain universal single qubit rotations, and it can also be extended to multi-qubit operations to yield universal quantum computation in the so-called measurement based model.

(c) **Other models of quantum computation**

(0.5 pt.) We have seen in the course a computationally equivalent model of quantum computation: the circuit model. What is a universal gate set and why is it useful? Give an example of universal gate set.