Quantum Optics and Quantum Informatics FKA173

Date and time: Tuesday, 24 October 2017, 08:30-12:30.
Examiners: Jonas Bylander (0702-53 44 39) and Thilo Bauch (0733-66 13 79).
Visits around 09:30 and 11:30.

1. **Bloch sphere manipulations (1.5 pts.)**

   (a) Write down a general qubit state using the angles θ and ϕ and depict it on the Bloch sphere. (0.5 pt.)

   (b) Given two time-dependent fields $B_x(t)$ and $B_y(t)$ that couple to a qubit, i.e. the Hamiltonian is given by

   $$\hat{H} = -\frac{\hbar\gamma}{2}\left(B_x(t)\widehat{\sigma}_x + B_y(t)\widehat{\sigma}_y\right),$$

   with $\gamma > 0$, describe how to take the qubit from the state $(|0\rangle + |1\rangle)/\sqrt{2}$ to the state $(|0\rangle - i\,|1\rangle)/\sqrt{2}$. During how long time should the fields be applied? (1 pt.)
   Note: You can use the "left hand rule" together with the time-dependent rotation angle $\delta = \gamma\int_0^t dt'|\mathbf{B}|(t')$ around the field axis. Keep in mind that you can change the fields along x and y direction independently.

2. **Rabi Hamiltonian and Jaynes-Cummings Hamiltonian (5.5 pts.)**

   (a) The Rabi Hamiltonian of an atom in presence of an external electromagnetic field is given by

   $$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + V(r) + e\hat{\mathbf{r}}\cdot\mathbf{E}(t) = \hat{H}_0 - \hat{\mathbf{d}}\cdot\mathbf{E}(t),$$

   where $\hat{\mathbf{d}} = -e\hat{\mathbf{r}}$ is the dipole operator. What approximation is used to derive this Hamiltonian? (0.5 pt.)

   (b) The Jaynes-Cummings Hamiltonian can be derived from the Rabi Hamiltonian by replacing the classical field $\mathbf{E}(t)$ of a single mode in a one dimensional cavity (resonator) by an operator $\hat{\mathbf{E}}(t) = \hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz)$, resulting in

   $$\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-).$$

   - What are the meanings of the three terms in the Hamiltonian? (0.5 pt.)
   - What approximation was used to derive this Hamiltonian and what is the physical meaning of this approximation? (0.5 pt.)

- In the two-dimensional subspace spanned by $|g\rangle|n{+}1\rangle$ and $|e\rangle|n\rangle$, the Jaynes-Cummings Hamiltonian looks like

$$\hat{H}_n = \hbar\omega\left(n + \frac{1}{2}\right)\hat{1} - \frac{\hbar(\omega_0 - \omega)}{2}\sigma_z + g\sqrt{n+1}\sigma_x,$$

  where $|e\rangle$ and $|g\rangle$ are respectively the excited and ground state of the atom, and $|n\rangle$ is the photon number state. Here the Pauli operators and the unity matrix $\hat{1}$ are represented in the $|g\rangle|n{+}1\rangle$ and $|e\rangle|n\rangle$ basis. Derive the Eigenenergies of the Hamiltonian for zero detuning $\omega = \omega_0$ and depict the energy spectrum of the uncoupled ($g = 0$) and coupled ($g \neq 0$) atom-photon states. Put the relevant energy differences into the drawings. (1 pt.)
  Note: The eigenvalues $\lambda^{\pm}$ of a 2×2 matrix can be found from

$$\left(a_{11} - \lambda^{\pm}\right)\left(a_{22} - \lambda^{\pm}\right) - a_{12}a_{21} = 0,$$

  where a_{11}, a_{22} are the diagonal and a_{12}, a_{21} the off-diagonal elements of the 2×2 matrix.

- Depict the two eigenstates of the above Hamiltonian on the Bloch sphere for zero detuning $\omega = \omega_0$ and $g > 0$. Depict the coupling energy $2g\sqrt{n+1}$ on the Bloch sphere as well. (0.5 pt.)

(c) For large detuning between the field frequency and the atomic transition frequency ($\Delta = \omega_0 - \omega,\ \hbar|\Delta| \gg g$) we obtain the dispersive Hamiltonian:

$$\hat{H}_{\text{disp}} = -\frac{1}{2}\left(\hbar\omega_0 + \frac{g^2}{\hbar\Delta}\right)\sigma_z + \left(\hbar\omega - \frac{g^2}{\hbar\Delta}\sigma_z\right)\hat{a}^{\dagger}\hat{a},$$

where σ_z is represented in the atom basis.

- What is the effective frequency of the atom when there are n photons in the cavity? (0.5 pt.)
- The dispersive Hamiltonian can be used to read out the atom/qubit state using coherent (microwave) photons. Explain explicitly how such a measurement can be implemented, what is measured, and how the qubit state can be detected. (1 pt.)
- Describe the nature of the back-action on the atom/qubit during this dispersive readout. Motivate your answer with reference to the dispersive Hamiltonian. (1 pt.)

3. **Single-qubit density matrix; Ramsey free-induction decay (2 pts.)**

Consider an ensemble of spin-1/2 systems quantized by a magnetic field along $\hat{z}$ in the laboratory frame of reference. Assume that the Larmor frequency is $\omega_0 \gg k_{\text{B}}T/\hbar$, where T is the temperature, k_{B} is Boltzmann's constant, and $\hbar$ is Planck's constant.

Let us assume that all spins are initially aligned with the field, i.e. in their $|0\rangle_z$ eigenstate. Then, let a $\pi/2|_x$ pulse bring the spins onto the $\hat{x} - \hat{y}$ plane. After a while, we measure the spin polarization along the three spatial directions and find the following average projections:

$$\langle\hat{\sigma}_x\rangle = r_x \qquad \langle\hat{\sigma}_y\rangle = r_y \qquad \langle\hat{\sigma}_z\rangle = r_z = 0$$

(a) Determine the density matrix ρ and represent it in the Bloch sphere.

(b) For $|r_x| = |r_y| = 1/2$, does this density matrix characterize a pure quantum state or a mixed state? What are the conditions on r_x and r_y (and r_z) for the ensemble to be in a pure state?

(c) What will r_x, r_y, and r_z be after a long time ($t \gg T_2$), assuming that the different spins that constitute the ensemble are subject to different (low-frequency noise) fields? Assume that there is no high-frequency noise present.

(d) Now assume that there is also high-frequency noise, transverse to the quantization axis (i.e. along $\hat{x}$ and/or $\hat{y}$). Then what will r_x, r_y, and r_z be after a long time ($t \gg T_1$)?

4. **Quantizing Electrical Circuits (3.5 pts.)**

Derive the quantum mechanical Hamiltonian of the single Cooper-pair box, starting from the electrical circuit in the figure 2. Here the tunnel junction is a parallel combination of a capacitance C_J and a Josephson junction with flux dependent potential energy $E_{JJ} = -E_J \cos\left(2\pi\frac{\Phi}{\Phi_0}\right)$, where E_J is a constant and Φ the flux across the junction and $\Phi_0 = h/(2e)$ the superconducting flux quantum.

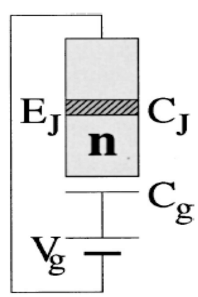

Figure 1: Circuit diagram of the single Cooper-pair box

(a) Write down the classical Lagrangian $\mathcal{L}(\Phi, \dot{\Phi})$ for the circuit. (1.5 pts.)
Note: First write the Lagrangian in branch flux representation and then rewrite the Lagrangian in node flux representation making use of Kirchhoff's relations (constraints).

(b) Find the conjugate momentum to the coordinate you chose. What is the physical meaning of the conjugate momentum? (0.5 pt.)
Note: The conjugate momentum of the flux Φ is given by $q = \partial\mathcal{L}/\partial\dot{\Phi}$.

(c) Write down the classical Hamiltonian for the circuit. (0.5 pt.)
Note: Use the Legendre transformation to obtain the Hamiltonian: $H(\Phi_n, q_n) = \sum_n \dot{\Phi}_n q_n - \mathcal{L}$

(d) Going to quantum mechanics, what are the commutation relations between all the operators in the Hamiltonian? (0.5 pt.)

(e) The quantized Hamiltonian, expressed in charge basis, is given by:

$$\hat{H} = \sum_n 4E_C(n - n_g)^2 |n\rangle\langle n| - \frac{E_J}{2}(|n\rangle\langle n+1| + |n+1\rangle\langle n|),$$

with $E_C = e^2/(2C_\Sigma)$ the Coulomb energy of a single electron on the island and $C_\Sigma = C_J + C_g$ the island capacitance to ground.
Truncate the Hamiltonian to a two-level system and write it using Pauli matrices. (0.5 pt.)

5. **Superconducting qubit experiment (3 pts.)**

In an elegant experiment, Hofheinz et al. generated and analyzed Fock states (number states) and coherent states in a superconducting resonator, using a superconducting qubit.[1] The dynamics of energy exchange betwen the resonator and the qubit near resonance can be approximated by the Jaynes–Cummings model Hamiltonian [cf. question 2(b)], where we now write $g = \hbar\Omega/2$,

$$\hat{H}_{\text{interaction}} = \frac{\hbar\Omega}{2}(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-). \tag{1}$$

Hofheinz loaded Fock states into the resonator by using a superconducting qubit that could be brought in and out of resonance with the resonator.

(a) How can this frequency tuning be done with a superconducting qubit, in the laboratory? How can manipulation pulses (gates) be experimentally applied to the qubit? Give an example. (0.5 pt.)

Starting with an initially empty resonator and the qubit in its ground state, $|g, 0\rangle$, the following program loads n photons into the resonator.

i. With the qubit and resonator far detuned in frequency, do a π pulse on the qubit: $|g, 0\rangle \to |e, 0\rangle$.
ii. Bring the two systems into resonance for a time τ_1, adjusted to swap the excitation from the qubit to the resonator: $|e, 0\rangle \to |g, 1\rangle$.

[1] Hofheinz et al., Nature vol. 454, p. 310–314 (2008).

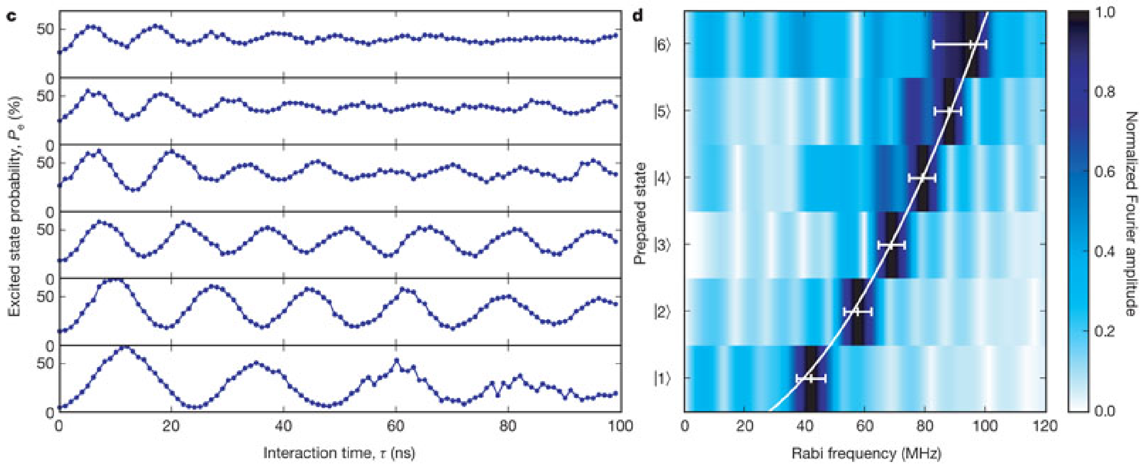

Figure 2: Data from Hofheinz et al. Note that the experimental measurement fidelity is less than 100 %; therefore, the oscillations in $P_{\mathrm{e}}(\tau)$ do not go from 0 % (corresponding to the qubit's ground state) to 100 % (excited state).

iii. Bring the systems out of resonance again.
iv. Do a π pulse on the qubit: $|g,1\rangle \to |e,1\rangle$.
v. Bring the systems into resonance for a time τ_2: $|e,1\rangle \to |g,2\rangle$.
vi. Repeat until the desired state $|g,n\rangle$ has been obtained.

(b) How long should the interaction times τ_k be for $k = 1, 2, \ldots, n$? (1 pt.)

In order to analyze the state of the resonator, after loading it with photons, Hofheinz brought the qubit (in its ground state) into resonance, now for a time τ, and then out of resonance again. He then measured the probability to find the qubit in its excited state vs. the time of interaction, $P_{\mathrm{e}}(\tau)$. The figure shows the resulting data.

(c) Explain this data. Which states of the combined system are involved in the oscillations? Why is the frequency different, depending on which Fock state $|n\rangle$ had been prepared? Can you comment on the similarity to stimulated emission? (1.5 pts.)

6. **Quantum circuits and algorithms; entanglement (4.5 pts.)**

(a) What does it mean that a quantum algorithm can solve certain problems efficiently? (0.5 pt.)

(b) What is an entangling gate? What can entangled states be used for? Give one example of a quantum circuit that represents a two-qubit entangling gate and write down its matrix representation. (1 pt.)

(c) Compute the action of the gate in question (b) for a product-state input that entangles the two qubits. Demonstrate that the output state is indeed entangled. (2 pts.)

(d) Give one example of how an entangling gate can be implemented in a physical system. (1 pt.)