Quantum Optics and Quantum Informatics FKA173

Date and time: Tuesday, 27 October 2015, 08:30-12:30.
Examiners: Jonas Bylander (0702-53 44 39) and Thilo Bauch (0733-66 13 79).
Visits around 09:30 and 11:30.

1. **Bloch sphere manipulations** (1.5 pts.)

   (a) Write down a general qubit state and depict it on the Bloch sphere. (0.5)

   (b) Given two time-dependent fields $B_z(t)$ and $B_x(t)$ that couple to a qubit, i.e. the Hamiltonian is given by

   $$\hat{H} = -\frac{1}{2}\left(B_x(t)\widehat{\sigma}_x + B_z(t)\widehat{\sigma}_z\right),$$

   describe how to take the qubit from the state $|1\rangle$ to the state $\left(|0\rangle + |1\rangle\right)/\sqrt{2}$. During how long time should the fields be applied?
   Note: You can use the "left hand rule" together with the time-dependent rotation angle $\delta = \frac{1}{\hbar}\int_0^t dt'|\mathbf{B}|(t')$ around the field axis. Keep in mind that you can change the fields along x and z direction independently. (1)

2. **Rabi Hamiltonian and Jaynes-Cummings Hamiltonian** (5.5 pts.)

   (a) The Rabi Hamiltonian of an atom in presence of an external electromagnetic field is given by

   $$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + V(r) + e\hat{\mathbf{r}} \cdot \mathbf{E}(t) = \hat{H}_0 - \hat{\mathbf{d}} \cdot \mathbf{E}(t),$$

   where $\hat{\mathbf{d}} = -e\hat{\mathbf{r}}$ is the dipole operator. What approximation is used to derive this Hamiltonian? (0.5)

   (b) The Jaynes-Cummings Hamiltonian can be derived from the Rabi Hamiltonian by replacing the classical field $\mathbf{E}(t)$ of a single mode in a one dimensional cavity (resonator) by an operator $\hat{\mathbf{E}}(t) = \hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin{(kz)}$, resulting in

   $$\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-).$$

- What are the meanings of the three terms in the Hamiltonian? (0.5)
- What approximation was used to derive this Hamiltonian and what is the physical meaning of this approximation? (0.5)
- In the two-dimensional subspace spanned by $|e\rangle|n\rangle$ and $|g\rangle|n+1\rangle$, the Jaynes-Cummings Hamiltonian looks like

$$\hat{H}_n = \hbar\omega\left(n + \frac{1}{2}\right)\hat{1} - \frac{\hbar(\omega_0 - \omega)}{2}\sigma_z + g\sqrt{n+1}\sigma_x,$$

  where $|e\rangle$ and $|g\rangle$ are respectively the excited and ground state of the atom, and $|n\rangle$ is the photon number state. Derive the Eigenenergies of the Hamiltonian for zero detuning $\omega = \omega_0$ and depict the energy spectrum of the uncoupled ($g = 0$) and coupled ($g \neq 0$) atom-photon states. Put the relevant energy differences into the drawings.
  Note: The eigenvalues $\lambda^{\pm}$ of a 2×2 matrix can be found from

$$\left(a_{11} - \lambda^{\pm}\right)\left(a_{22} - \lambda^{\pm}\right) - a_{12}a_{21} = 0,$$

  where a_{11}, a_{22} are the diagonal and a_{12}, a_{21} the off-diagonal elements of the 2×2 matrix. (1.5)

(c) For large detuning ($\Delta = \omega_0 - \omega,\ \hbar|\Delta| \gg g$) between the field frequency and the atomic transition frequency we obtain the dispersive Hamiltonian:

$$\hat{H}_{\text{disp}} = -\frac{1}{2}\left(\hbar\omega_0 + \frac{g^2}{\hbar\Delta}\right)\sigma_z + \left(\hbar\omega - \frac{g^2}{\hbar\Delta}\sigma_z\right)\hat{a}^\dagger\hat{a}.$$

- What is the effective frequency of the cavity when the atom (the qubit) is in the ground state? (0.5)
- This system can be used to read out the qubit state using (microwave) photons. Explain explicitly how the system should be modified, what is measured and how the qubit state can be detected. (1)
- Describe the nature of the back-action on the qubit during this dispersive readout. Motivate your answer with reference to the dispersive Hamiltonian. (1)

3. **Ramsey fringes, mixed states and the Bloch equations** (5 pts.)

Let a qubit start out in the ground state, and apply a sequence of two $\pi/2$ pulses separated by a time τ of free induction, followed by a projective measurement onto the energy eigenbasis. The lab-frame Hamiltonian is the same as that in question 1. Figure 1 shows the probability to find a qubit in the ground state after this sequence (Ramsey fringes).

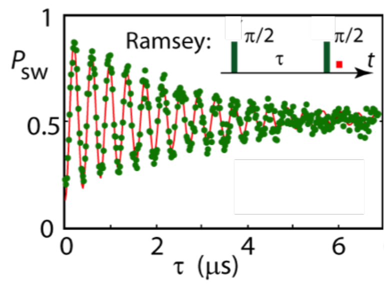

Figure 1: Ramsey interference fringes (superconducting qubit). P_{sw} denotes the probability to find the qubit in its ground state when measuring after the second pulse (indicated by the red dot).

(a) Explain why the signal decays toward 0.5. What is the relevant characteristic time scale usually called, and how long is this characteristic time here? Explain why the signal does not initially swing between 0 and +1, but only between approximately 0.2 and 0.9. (2)

(b) Write down the Bloch equations for a spin-1/2,

$$\dot{\mathbf{r}} = \mathbf{r} \times \mathbf{B}(t) - \frac{r_z - r_0}{T_1}\hat{z} - \frac{r_x\hat{x} + r_y\hat{y}}{T_2},$$

for the average projections onto the x, y, and z axes (Bloch vector), starting right after the first $\pi/2$ pulse, i.e. at time $t_{\pi/2}$ when the qubit state is on the equator. You can assume that the pulse was along the x axis, so that the state points along the y axis at this moment in time.

Solve the Bloch equations for the time τ of free-induction decay, i.e. until right before the second $\pi/2$ pulse. Illustrate your results in the Bloch sphere. (You can make an *Ansatz* for the form of the solution and then verify it. It's ok to assume resonant pulses.) (2)

(c) Write down the density matrix for the times (i) before the first $\pi/2$ pulse; (ii) after the first $\pi/2$ pulse; and (iii) at time τ. Is the state pure or mixed at the times (i–iii)? (1)

4. **Quantizing Electrical Circuits** (3.5 pts.)

Derive the quantum mechanical Hamiltonian of the single Cooper-pair box, starting from the electrical circuit in the figure 2. Here the tunnel junction is a parallel combination of a capacitance C_J and a Josephson junction with flux dependent potential energy $E_{JJ} = -E_J \cos\left(2\pi \frac{\Phi}{\Phi_0}\right)$, where E_J is a constant and Φ the flux across the junction and $\Phi_0 = h/(2e)$ the superconducting flux quantum.

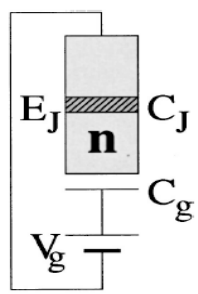

Figure 2: Circuit diagram of the single Cooper-pair box

(a) Write down the classical Lagrangian $\mathcal{L}(\Phi, \dot{\Phi})$ for the circuit.
Note: First write the Lagrangian, i.e. kinetic energy minus potential energy, in branch flux representation and then rewrite the Lagrangian in node flux representation making use of Kirchhoff's relations (constraints).(1.5)

(b) Find the conjugate momentum to the coordinate you chose, and motivate the choice of coordinate.
Note: The conjugate momentum of the flux Φ is given by the charge $q = \partial\mathcal{L}/\partial\dot{\Phi}$. (0.5)

(c) Write down the classical Hamiltonian for the circuit.
Note: Use the Legendre transformation to obtain the Hamiltonian: $H(\Phi_n, q_n) = \sum_n \dot{\Phi}_n q_n - \mathcal{L}$ (0.5)

(d) Going to quantum mechanics, what are the commutation relations between all the operators in the Hamiltonian? (0.5)

(e) The quantized Hamiltonian, expressed in charge basis, is given by:

$$\hat{H} = \sum_n 4E_C(n - n_g)^2 |n\rangle\langle n| - \frac{E_J}{2}(|n\rangle\langle n+1| + |n+1\rangle\langle n|),$$

with $E_C = e^2/(2C_\Sigma)$ the Coulomb energy of a single electron on the island and $C_\Sigma = C_J + C_g$ the island capacitance to ground.
Truncate the Hamiltonian to a two-level system and write it using Pauli matrices. (0.5)

5. **Quantum circuits and algorithms** (6 pts.)

(a) Show that for the controlled-Z operation on two qubits (see Fig. 3), the outcome is the same regardless of which qubit controls which. (1)

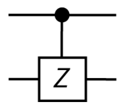

Figure 3: Controlled-Z gate.

(b) Construct a CNOT gate (see Fig. 4) from one controlled-Z gate and two Hadamard gates, specifying control and target qubits. (1)

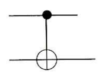

Figure 4: Controlled-NOT gate.

(c) The quantum teleportion circuit is shown in Fig. 5. Explain what happens throughout this algorithm. Explicitely write out the states $|\Psi_0\rangle$, $|\Psi_1\rangle$, and $|\Psi_2\rangle$. Explain how Bob obtains the output state, based on the measurement results m and n. (2.5)

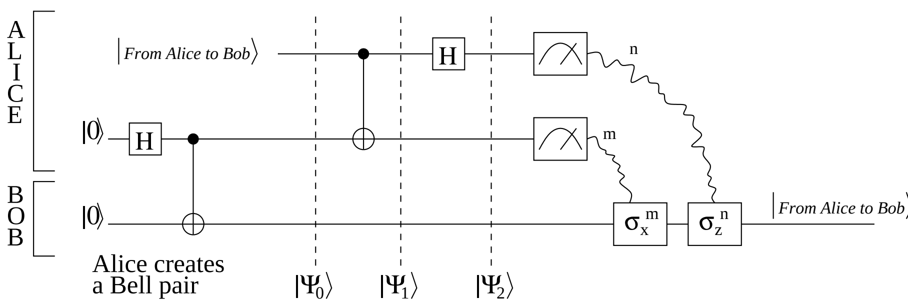

Figure 5: Quantum teleportation circuit.

(d) The Bell states are the "maximally entangled" two-qubit states. Explain what this means. Demonstrate that the CNOT is an entangling gate by calculating the reduced density matrix for one of the qubits in a Bell pair $|\Psi\rangle$,

$$\hat{\rho}_1 = \mathrm{Tr}(\hat{\rho}_2) = \sum_{j=0}^{1} \langle j_2|\Psi\rangle\langle\Psi|j_2\rangle.$$

(1.5)