Quantum Informatics FKA172

Time: Saturday 14th of January 2012 at 08.30-12.30.
Examiner:
Göran Johansson (mob: 0730-607338, visits around 09:30 and 11:30).
Allowed reference: A hand written A4 sheet (both sides may be filled).
Grading: There are six questions. The points awarded for each question is indicated by that question and in the case of subquestions the division of the points is also indicated. The total number of points is 18.

1. **Bloch sphere manipulations** (2 pts.)

   a) Write down a general qubit state and depict it on the Bloch sphere. (0.5)
   b) Given two fields $B_z(t)\widehat{\sigma}_z$ and $B_x(t)\widehat{\sigma}_x$ that couple to a qubit, describe how to take the state $|0\rangle$ to the state $(|0\rangle + |1\rangle)/\sqrt{2}$. During how long time should the fields be applied? (1.5)

2. **Rabi Hamiltonian and Jaynes-Cummings Hamiltonian** (5 pts.)

   a) The Rabi Hamiltonian of an atom in presence of an external electromagnetic field is given by

   $$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + V(r) + e\hat{\mathbf{r}} \cdot \mathbf{E}(t) = \hat{H}_0 - \hat{\mathbf{d}} \cdot \mathbf{E}(t),$$

   where $\hat{\mathbf{d}} = -e\hat{\mathbf{r}}$ is the dipole operator. What approximation is used to derive this Hamiltonian? (0.5)

   b) The Jaynes-Cummings Hamiltonian can be derived from the Rabi Hamiltonian by replacing the classical field $\mathbf{E}(t)$ of a single mode in a one dimensional cavity (resonator) by an operator $\hat{\mathbf{E}}(t) = \hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz)$, resulting in:

   $$\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-).$$

   - What is the meaning of the three terms in the Hamiltonian? (0.5)
   - What approximation was used to derive this Hamiltonian and what is the physical meaning of this approximation? (0.5)
   - Depict the energy spectrum of the uncoupled ($g = 0$) and coupled ($g \neq 0$) atom-photon states for zero detuning $\omega = \omega_0$. Put the relevant energy differences into the drawings. (1)

   c) For large detuning ($\Delta = \omega_0 - \omega,\ \hbar|\Delta| \gg g$) between the field frequency and atom transition frequency we obtain the dispersive Hamiltonian:

   $$\hat{H}_{\text{disp}} = -\frac{1}{2}\left(\hbar\omega_0 + \frac{g^2}{\hbar\Delta}\right)\sigma_z + \left(\hbar\omega - \frac{g^2}{\hbar\Delta}\sigma_z\right)\hat{a}^\dagger\hat{a}.$$

- What is the effective frequency of the cavity when the atom (qubit) is in the ground state? (0.5)
- This system can be used to read out the qubit state using (microwave) photons. Explain explicitly how the system should be modified, what is measured and how the qubit state can be detected. (1.5)
- Describe the nature of the back-action on the qubit during this dispersive readout. Motivate your answer with reference to the dispersive Hamiltonian. (1)

3. **Mixed states and the Bloch equations** (2.5 pts.)

   Calculate the density matrix after each step a)-e), for a qubit going through the following consecutive evolutions:
   a) First being initialized in the ground state
   b) then after a $\pi/2$-rotation around the x-axis (0.5)
   c) then after being subject to pure dephasing (along the z-axis), characterized by the timescale T_2^*, for a time ΔT_φ. (0.5)
   d) then after a $-\pi/2$-rotation around the x-axis (0.5)
   e) then after being subject to relaxation characterized by the time-scale T_1, by coupling to an environment at *zero temperature*, for a time of ΔT_r. (0.5)

   Finally, what is the probability to find the qubit in the excited state (in terms of the ratios $\Delta T_\varphi/T_2^*$ and $\Delta T_r/T_1$)? (0.5)

4. **Quantizing Electrical Circuits** (3 pts.)
   Derive the quantum mechanical Hamiltonian of the single Cooper-pair box, starting from the electrical circuit.

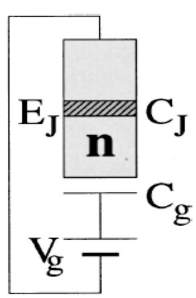

Figure 1: The circuit diagram of the single Cooper-pair box

   a) Write down the classical Lagrangian for the circuit. (0.5)
   b) Find the conjugate momentum to the coordinate you chose, and motivate the choice of coordinate. (0.5)

c) Write down the classical Hamiltonian for the circuit. (Note, the Hamiltonian is written in terms of the conjugate momentun, i.e. *not* the velocity.) (0.5)
d) Going to quantum mechanics, what are the commutation relations between all the operators in the Hamiltonian? (0.5)
e) Truncate the Hamiltonian to a two-level system and write it using Pauli matrices. (0.5)
f) Briefly discuss how to include resistors and other dissipative elements in a quantum mechanical circuit. (0.5)

5. **The No-Cloning Theorem** (1.5pts.)
   State and prove the no-cloning theorem.

6. **Quantum Error Correction** (4 pts.)

   The circuit in Fig. 2 below implements quantum error correction using a three-qubit code.
   1) The "Bit Flip" box randomly applies an X-gate to *at most* one qubit (could also be none). Describe the error correction properties of the circuit gate by gate, for all possible (allowed) actions of the "Bit Flip" box. (2.5 pts.)
   2) Show explicitly that if the "Bit Flip" box would do something else, like introducing a phase-flip (Z-gate) to a qubit, the error correction would not work. (1 pt.)
   3) Is it possible to correct for a general single qubit error? If yes, motivate what size of code would be needed? (0.5 pts.)

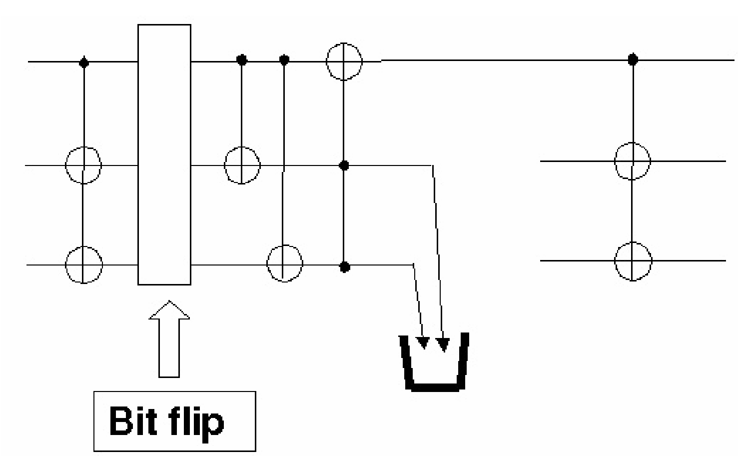

Figure 2: A circuit implementing a three qubit error correction code. The "waste-basket" illustrates the reset of the lower two qubits.