Quantum Informatics FKA172

Time: Wednesday 19th of October 2011 at 14.00-18.00.
Examiner:
Göran Johansson (mob: 0730-607338, visits around 15:15 and 17:00).
Allowed reference: A hand written A4 sheet (both sides may be filled).
Grading: There are eight questions. The points awarded for each question is indicated by that question and in the case of subquestions the division of the points is also indicated. The total number of points is 21.

1. **Bloch sphere manipulations** (2 pts.)

   Given two fields $B_z(t)\widehat{\sigma}_z$ and $B_x(t)\widehat{\sigma}_x$ that couple to a qubit, describe how to take the state $|0\rangle$ to the state $(|0\rangle - |1\rangle)/\sqrt{2}$. During how long time should the fields be applied?

2. **Rabi Hamiltonian and Jaynes-Cummings Hamiltonian** (5.5 pts.)

   a) The Rabi Hamiltonian of an atom in presence of an external electromagnetic field is given by

   $$\hat{H} = \frac{\hat{\mathbf{P}}^2}{2m} + V(r) + e\hat{\mathbf{r}} \cdot \mathbf{E}(t) = \hat{H}_0 - \hat{\mathbf{d}} \cdot \mathbf{E}(t),$$

   where $\hat{\mathbf{d}} = -e\hat{\mathbf{r}}$ is the dipole operator. What approximation is used to derive this Hamiltonian? (0.5)

   b) The Jaynes-Cummings Hamiltonian can be derived from the Rabi Hamiltonian by replacing the classical field $\mathbf{E}(t)$ of a single mode in a one dimensional cavity (resonator) by an operator $\hat{\mathbf{E}}(t) = \hat{E}_x(z,t) = \mathcal{E}_0(\hat{a} + \hat{a}^\dagger)\sin(kz)$, resulting in:

   $$\hat{H} = -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\omega\hat{a}^\dagger\hat{a} + g(\hat{a}\sigma_+ + \hat{a}^\dagger\sigma_-).$$

   - What is the meaning of the three terms in the Hamiltonian? (1.5)
   - What approximation was used to derive this Hamiltonian and what is the physical meaning of this approximation? (0.5)
   - Depict the energy spectrum of the uncoupled ($g = 0$) and coupled ($g \neq 0$) atom-photon states for zero detuning $\omega = \omega_0$. Put the relevant energy differences into the drawings. (1)

   c) For large detuning ($\Delta = \omega_0 - \omega,\ \hbar|\Delta| \gg g$) between the field frequency and atom transition frequency we obtain the dispersive Hamiltonian:

   $$\hat{H}_{\text{disp}} = -\frac{1}{2}\left(\hbar\omega_0 + \frac{g^2}{\hbar\Delta}\right)\sigma_z + \left(\hbar\omega - \frac{g^2}{\hbar\Delta}\sigma_z\right)\hat{a}^\dagger\hat{a}.$$

- What is the effective frequency of the cavity when the atom (qubit) is in the excited state? (0.5)
- How would you read out the (atom) qubit state, and would there be any back action on the qubit during readout? (1.5)

3. **Mixed states and the Bloch equations** (2.5 pts.)

   Calculate the density matrix after each step a)-e), for a qubit going through the following consecutive evolutions:
   a) First being initialized in the ground state
   b) then after a $\pi/2$-rotation around the x-axis (0.5)
   c) then after being subject to pure dephasing (along the z-axis), characterized by the timescale T_2^*, for a time ΔT_φ. (0.5)
   d) then after a $-\pi/2$-rotation around the x-axis (0.5)
   e) then after being subject to relaxation characterized by the time-scale T_1, by coupling to an environment at *zero temperature*, for a time of ΔT_r. (0.5)

   Finally, what is the probability to find the qubit in the excited state (in terms of the ratios $\Delta T_\varphi / T_2^*$ and $\Delta T_r / T_1$)? (0.5)

4. **Quantizing Electrical Circuits** (3 pts.)
   Derive the quantum mechanical Hamiltonian of the single Cooper-pair box, starting from the electrical circuit.

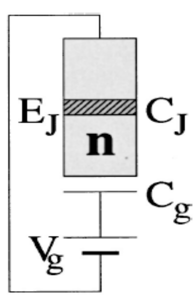

Figure 1: The circuit diagram of the single Cooper-pair box

   a) Write down the classical Lagrangian for the circuit. (0.5)
   b) Find the conjugate momentum to the coordinate you chose. (0.5)
   c) Write down the classical Hamiltonian for the circuit. (0.5)
   d) Going to quantum mechanics, what are the commutation relations between the operators? (0.5)
   e) Truncate the Hamiltonian to a two-level system. (0.5)

f) Briefly discuss how to include resistors and other dissipative elements in a quantum mechanical circuit. (0.5)

5. **The CNOT gate** (1 pt.)

   For the CNOT gate write down
   1) the circuit symbol
   2) the truth table, i.e. the output states for the different input basis states
   3) the corresponding unitary matrix

6. **Quantum Teleportation** (2 pts.) Explain the concept of quantum teleportation, including the initial setup, the people involved and their resources, the main steps and what is actually teleported. Can quantum teleportation be used to transmit information faster than the speed of light? Motivate your answer!

7. **The No-Cloning Theorem** (2pts.) Prove the no-cloning theorem (1.5), and discuss how it relates to teleportation. (0.5)

8. **Deutsch's Algorithm** (3 pts.)

   In Fig. 2 a circuit implementing Deutsch's algorithm is shown. What is the problem solved by this algorithm? How much faster is this algorithm compared to any classical algorithm for the same problem? Show explicitly how the algorithm works, i.e. write down the states $|\Psi_0\rangle$, $|\Psi_1\rangle$, $|\Psi_2\rangle$ and $|\Psi_3\rangle$, and specify what measurement should be performed and how the result answers the problem.

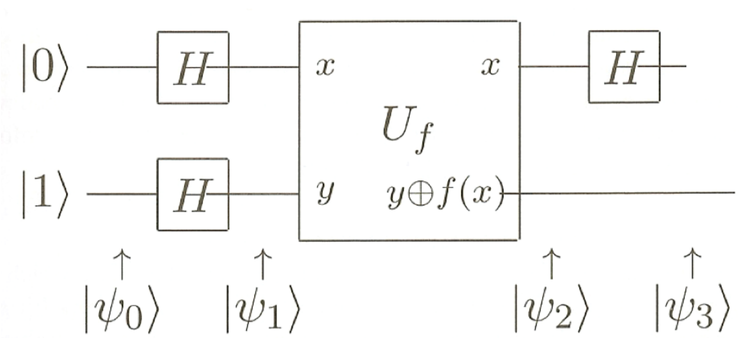

Figure 2: A circuit implementing Deutsch's algorithm.