Stochastic optimization methods (FFR 105), 2022
Solutions to the exam (2022-10-26)

1. 1.1 D. Only velocities should be restricted.
   1.2 B. Both eigenvalues are positive, so the function is convex.
   1.3 B. The probability equals $p = (1 - p_{\text{tour}})p_{\text{tour}}$.
   1.4 D. The list of points found by applying the Lagrange multiplier method contains all optima of f subject to the constraint, both local and global ones.
   1.5 C. The stationary points are $x = -1$ and $x = 2$.
   1.6 E. The probability equals $p_2 = \frac{1}{25}\left(2\frac{3}{4} + 6\frac{1}{4} + 1\right) = 0.16$.
   1.7 A. Any number of genes (in the range $[0, m]$) can mutate, in principle, since the mutation is applied on a gene-by-gene basis.
   1.8 C. No general statement can be made regarding the symmetry of either matrix.
   1.9 A. An individuals with fitness 0 will not be selected with RWS, in which selection is directly proportional to the fitness values. However, it can be selected with TS, which only considers the fitness values (of the participants in the tournament) relative to each other.
   1.10 D. Since the pheromone levels are the same on all edges, the pheromone need not be considered. With $\beta = 1$, the probability of selecting node 3 equals

$$p = \frac{5}{1+2+5} = \frac{5}{8} \tag{1}$$

2. With f and h as specified in the problem formulation, L takes the form

$$L = x^2 + y^2 + z^2 + \lambda(x^4 + y^4 + z^4 - 1). \tag{2}$$

so that, setting the partial derivatives of L to 0, one obtains

$$\frac{\partial L}{\partial x} = 2x + 4\lambda x^3 = 0, \tag{3}$$

$$\frac{\partial L}{\partial y} = 2y + 4\lambda y^3 = 0, \tag{4}$$

$$\frac{\partial L}{\partial z} = 2z + 4\lambda z^3 = 0, \tag{5}$$

$$\frac{\partial L}{\partial \lambda} = x^4 + y^4 + z^4 - 1 = 0. \tag{6}$$

It is clear that λ cannot be equal to 0, since that would give $x = y = z = 0$ which violates the constraint. With $\lambda \neq 0$ one gets the following potential solutions for x

$$x = 0 \tag{7}$$

or

$$x^2 = -\frac{1}{2\lambda}, \tag{8}$$

and similar potential solutions for y and z. First, consider the case where two of the variables, say x and y, take the value 0. In that case, one obtains (using the constraint equation)

$$z = \pm 1. \tag{9}$$

so that the following stationary points are found

$$P_{1,2} = (0, 0, \pm 1). \tag{10}$$

Using the fact that the problem is symmetric in the three variables, one also obtains

$$P_{3,4} = (\pm 1, 0, 0) \tag{11}$$

if $y = z = 0$, and

$$P_{5,6} = (0, \pm 1, 0) \tag{12}$$

if $x = z = 0$. Next, consider the case where one of the variables takes the value 0. Starting, for example, with $x = 0$ and using the equations (see above)

$$y^2 = -\frac{1}{2\lambda} \tag{13}$$

and

$$z^2 = -\frac{1}{2\lambda}, \tag{14}$$

one gets $y^2 = z^2$ so that, $y = \pm z$. Using the constraint equation one then finds

$$y = \pm 2^{-1/4}, \tag{15}$$

thus giving the following stationary points of L

$$P_{7,8,9,10} = (0, \pm 2^{-1/4}, \pm 2^{-1/4}). \tag{16}$$

Using again the symmetry of the problem, one also finds additional stationary points at

$$P_{11,12,13,14} = (\pm 2^{-1/4}, 0, \pm 2^{-1/4}) \tag{17}$$

if $y = 0$ and

$$P_{15,16,17,18} = (\pm 2^{-1/4}, \pm 2^{-1/4}, 0) \tag{18}$$

for $z = 0$. Finally, consider the case where none of the variables takes the value zero. In that case

$$x^2 = y^2 = z^2 = -\frac{1}{2\lambda}, \tag{19}$$

so that the constraint gives $3x^4 = 1$, from which one obtains the following stationary points of L

$$P_{19,20} = (\pm 3^{-1/4}, \pm 3^{-1/4}, \pm 3^{-1/4}). \tag{20}$$

The values of the function f at these points are: $f = 1$ at $P_1, \ldots, P_6$, $f = \sqrt{2}$ at $P_7, \ldots, P_{18}$, and $f = \sqrt{3}$ at $P_{19,20}$. Thus, the minimum value of f is equal to 1, and is attained as $P_1, \ldots, P_6$, whereas the maximum value is $\sqrt{3}$, and is attained at $P_{19,20}$. Note: This problem can also be solved by substituting $x^2 = a$, $y^2 = b$ and $z^2 = c$, and then solving for (a, b, c), while making sure to include both possibilities (e.g. $x = \pm a$) when finding the stationary points.

3. Given the specification in the problem formulation, the instructions (here denoted $I_j, j = 1, \ldots 6$) can be decoded as

I_1	1216:	$r_2 := r_1 + c_3$
I_2	3322:	$r_3 := r_2 \times r_2$
I_3	1214:	$r_2 := r_1 + c_1$
I_4	1323:	$r_3 := r_2 + r_3$
I_5	1115:	$r_1 := r_1 + c_2$
I_6	1213:	$r_2 := r_1 + r_3$

Starting from $r_1 = x$, $r_2 = r_3 = 0$, one then gets

	r_1	r_2	r_3
I_1	x	$x - 1$	0
I_2	x	$x - 1$	$(x-1)^2$
I_3	x	$x + 1$	$(x-1)^2$
I_4	x	$x + 1$	$x + 1 + (x-1)^2$
I_5	$x + 2$	$x + 1$	$x + 1 + (x-1)^2$
I_6	$x + 2$	$x + 2 + x + 1 + (x-1)^2$	$x + 1 + (x-1)^2$

Hence, the function obtained (from r_2) at the end of the calculation is

$$g(x) = x + 2 + x + 1 + (x-1)^2 = 2x + 3 + x^2 - 2x + 1 = x^2 + 4. \tag{21}$$

4. The minimum of the simple quadratic function $f(x)$ is clearly at $x = 3/16$. Considering the decoding procedure specified in the problem, this corresponds to the chromosome 00110 (which will then be decoded to give $x = 2^{-3} + 2^{-4} = 3/16$). In order to generate this chromosome, an ant must traverse the construction graph such that it first makes two down-moves (at Nodes 1 and 4), and two up-moves (at Nodes 7 and 10), and then one down-move (at Node 13). Note that, at Nodes $2, 3, 5, 6, \ldots$, the move to the next node (i.e. $4, 7, \ldots$) is deterministic and does not produce any output. Consider now the tour of one ant. Use a simplified notation, such that $p(e_{ij}|S)$ is denoted $p_{i,j}$.

   At Node 1, the ant can either go to Node 2 or to Node 3. Noting that the visibility (η_{ij}) is equal to 1 for all edges, the probability of making the required down-move (i.e. going to Node 3, so as to generate $a_1 = 0$) can be computed as

$$p_{3,1} = \frac{\tau_{31}^{\alpha}}{\tau_{31}^{\alpha} + \tau_{21}^{\alpha}} = \frac{0.40}{0.40 + 0.35} = \frac{8}{15}, \tag{22}$$

   where, in the second step, the fact that $\alpha = 1$ has been used. The ant then moves from Node 2 to Node 4, in preparation for the next bit generation step. At Node 4, the probability of making a down-move (to output $a_2 = 0$) equals

$$p_{6,4} = \frac{0.30}{0.30 + 0.60} = \frac{1}{3}. \tag{23}$$

   Next, after moving to Node 7, the ant should then move to Node 8 (to yield $a_3 = 1$). The probability for this move equals

$$p_{8,7} = \frac{0.55}{0.55 + 0.40} = \frac{11}{19}. \tag{24}$$

   Then, after reaching Node 10, the ant should move to Node 11 (to generate $a_4 = 1$). The probability for this move equals

$$p_{11,10} = \frac{0.50}{0.50 + 0.65} = \frac{10}{23}. \tag{25}$$

   After going to Node 13, the ant should then move to Node 15, to yield $a_5 = 0$. This probability for making this move is

$$p_{15,13} = \frac{0.90}{0.20 + 0.90} = \frac{9}{11}. \tag{26}$$

Thus, the probability P of generating 001100 as output equals

$$P = p_{3,1} \times p_{6,4} \times p_{8,7} \times p_{11,10} \times p_{15,13} \approx 0.036613. \tag{27}$$

Now, the population consists of five ants that generate their paths independently of each other, and with the same pheromone levels, as specified in the problem formulation. For any given ant, the probability of *not* finding the required path is equal to $1 - P$. The probability that no ant finds this path thus equals $(1 - P)^5$ and therefore the probability Π that *at least* one ant finds the path is

$$\Pi = 1 - (1 - P)^5 = 0.1701, \tag{28}$$

and this is the answer.

5. (a) Initially, the function values are 49/144 (particle 1), 1/16 (particle 2), and 1/4 (particle 3). Thus, the swarm best position is equal to the position of particle 2 (i.e. $x = 0$). With the simplifications, the velocity update takes the form

$$v_i \leftarrow v_i + 2(x_i^{\text{pb}} - x_i) + 2(x^{\text{sb}} - x_i), \; i = 1, 2, 3. \tag{29}$$

One then obtains:

$$v_1 = 3 + 2(0 - (-1/3)) + 2(0 - (-1/3)) = 11/3, \tag{30}$$

$$v_2 = 1/4 + 2(0 - 0) + 2(0 - 0) = 1/4, \tag{31}$$

and

$$v_3 = -1 + 2(3/4 - 3/4) + 2(0 - 3/4) = -5/2. \tag{32}$$

All computed speed values have magnitudes below the limit ($v_{\max} = 4$). Thus, using the equation $x \leftarrow x + v$, the new positions become

$$x_1 = -1/3 + 11/3 = 10/3, \tag{33}$$

$$x_2 = 0 + 1/4 = 1/4, \tag{34}$$

$$x_3 = 3/4 - 5/2 = -7/4. \tag{35}$$

(b) In the second iteration, the swarm best position is $x = 1/4$, i.e. the position of particle 2 (which, of course, also is the particle best position for that particle). The particle best position is unchanged for particle 1 and particle 3, since the function values at their new positions exceed those obtained at their initial positions. Using the same equations as above, one obtains

$$v_1 = 11/3 + 2(-1/3 - 10/3) + 2(1/4 - 10/3) = -59/6. \quad (36)$$

However, this value exceeds (in magnitude) the maximum (negative) speed of -4, meaning that the actual speed of the particle will be $v_3 = -4$ instead. For particle 2 one gets

$$v_2 = 1/4 + 2(1/4 - 1/4) + 2(1/4 - 1/4) = 1/4 \quad (37)$$

and for particle 3

$$v_3 = -5/2 + 2(3/4 - (-7/4)) + 2(1/4 - (-7/4)) = 13/2. \quad (38)$$

This value is larger than the limit of 4, so that the actual speed will be $v_3 = 4$ instead. Thus, finally, one obtains

$$x_1 = 10/3 - 4 = -2/3, \quad (39)$$

$$x_2 = 1/4 + 1/4 = 1/2, \quad (40)$$

and

$$x_3 = -7/4 + 4 = 9/4. \quad (41)$$