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Exam in FFR 105 (Stochastic optimization algorithms), 2018-10-31, 14.00-18.00, M.

The examiner will visit the exam rooms twice, around 15.00 and around 17.00. It will be possible to review your results (exam and home problems) after Nov. 13.

In the exam, it is allowed to use a calculator, as long as it cannot store any text. Furthermore, mathematical tables (such as Beta, Standard Math etc.) are allowed, provided that no notes have been added. However, it is *not* allowed to use the course book, or any lecture notes from the course, during the exam.

Note! In problems involving computation, show *clearly* how you arrived at your answer, i.e. include intermediate steps etc. Only giving the answer will result in zero points on the problem in question. There are four problems in the exam, and the maximum number of points is 25.

1. (a) Particle swarm optimization (PSO) is a stochastic optimization based on swarming in biological systems, and somewhat similar to genetic algorithms.
   i. Swarming is a frequent phenomenon in nature. Describe at least two reasons why this phenomenon occurs. (1p)
   ii. Write down the equation for the velocity update in the standard PSO algorithm, and explain all parts of the equation in detail. Pay particular attention to the variable indices (where applicable). In the equation for the velocity update, one parameter is responsible for handling the trade-off between exploration and exploitation. Your description should include a clear explanation (with equations and parameter values) of how this trade-off is handled in PSO. (4p)

   (b) In optimization, convexity of the objective function is a desirable property. Determine whether or not the function

$$f(x_1, x_2) = 4x_1^2 + 2x_2^2 - 2x_1x_2 \tag{1}$$

   is convex. (1p)

   (c) Tournament selection is a common selection operator in evolutionary algorithms (EAs). Consider a case with a population size of five where, in a given generation, the fitness values are $F_1 = 3$, $F_2 = 6$, $F_3 = 7$, $F_4 = 9$ and $F_5 = 15$. Assuming a tournament size of two, what is the probability of selecting individual 4 (in a single selection step), assuming that the tournament selection parameter p_{tour} is equal to 0.8. Show clearly how you arrive at your answer. (2p)

2. (a) Newton-Raphson's method is an iterative method for finding local optima of a twice differentiable function. Use this method to find the minimum of the function $f(x) = 1 + x^6 - x^2$, starting from $x \equiv x_0 = 1$. First, write down (for this particular function) the expression for x_{j+1} as a function of x_j. Then iterate until the difference between two consecutive iterates (i.e. $|x_{j+1} - x_j|$) drops below 10^{-5}. Next, make a table with three columns: (1) the index $j+1$ of the iterate, (2) the corresponding value x_{j+1} and (3) the difference $|x_{j+1} - x_j|$, and insert the values obtained for $j = 0, 1, 2, \ldots$ Finally, *prove* that the point found is really a minimum of $f(x)$. (3p)

   (b) The Lagrange multiplier method is applicable to optimization problems involving equality constraints. Using this method, find the minimum and maximum values of the function

$$f(x_1, x_2) = 3x_1x_2 + 2 \tag{2}$$

   subject to the constraint

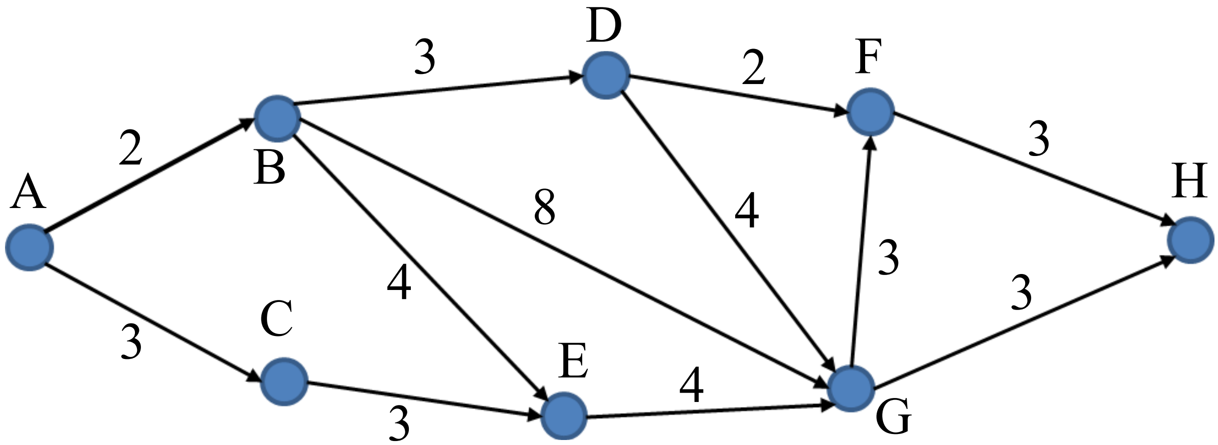
$$x_1^2 + x_2^2 = 1. \tag{3}$$

   In your answer, in addition to giving the minimum and maximum values, provide also the corresponding values of (x_1, x_2) for all points where the function takes either the minimum or the maximum value. (4p)

3. In analytical studies of genetic algorithms (GAs), it is common to use the Onemax problem, for which the value of the fitness function for a given (binary) chromosome equals the number of 1s in the chromosome. For this simple problem, one can derive an expression for the runtime and the optimal mutation rate for a GA with a single individual, which is modified using mutations only. In this GA, a mutated individual is kept if and only if it is better (i.e. its chromosome contains more 1s) than the previous individual.

   (a) Consider a chromosome of length m with l 0s (and, therefore, $m-l$ 1s). Let the mutation rate be p_{mut}. Derive an approximate expression for the probability of improving this chromosome (i.e. increasing the number of 1s). The expression should summarize a case in which none of the 1s mutate, and at least one of the 0s does. (1p)

   (b) Setting the mutation rate p_{mut} to k/m, where m is the chromosome length, and $k \ll m$ is a positive integer, *derive* an estimate for the runtime of this GA, i.e. the number of iterations required to reach the global optimum. Motivate clearly any approximations made in the derivation. (4p)



4. Ant colony optimization (ACO) can be used in vehicle routing problems. Consider the road network shown in the figure above. The network, which is not fully connected, consists of eight nodes and a number of directed edges. For simplicity, the edges are one-way in this problem. Thus, for example, there is an edge $e_{\mathrm{B} \leftarrow \mathrm{A}}$ from node A to node B, but no edge $e_{\mathrm{A} \leftarrow \mathrm{B}}$ from node B to node A etc. The numbers given on the edges are the measured traversal times (in some suitable time unit) between the two nodes connected by the edge in question. The traversal time depends both on the distance (not shown) between the two points and the amount of traffic.

   In this problem it is assumed that all vehicles start at node A and should move to node H, using probabilistic path generation as in the standard Ant system (AS) algorithm. Thus, it is possible for a vehicle to select a nominally slower route, perhaps to explore whether the traffic situation might have changed. (The traversal times are based on estimates from preceding days).

   In this case, the visibility of an edge is equal to the inverse of the traversal time for that edge. The objective function is taken as the inverse of the *total* traversal time as a vehicle moves from its start node to its end node. It is assumed that pheromone updates take place as soon as any vehicle reaches its end node.

   The parameters are as follows, using standard AS notation: $\alpha = 1$, $\beta = 2$, and $\rho = 0.5$, and the initial pheromone levels are set to $\tau_0 = 0.1$ for all edges.

   (a) What is the probability that the first vehicle will follow the fastest path from node A to node H (assuming that the estimated travel times are accurate for all edges)? (2p)

   (b) Assuming that the first vehicle actually *does* follow the fastest path, compute the updated pheromone levels on all edges (using the standard AS method for updating pheromones). (1p)

   (c) Assuming that the traversal times have not changed, what is the probability that the *next* vehicle will follow the fastest path from node A to node H, taking into account the changed pheromone levels resulting from the traversal of the first vehicle? (2p)