ESS101 Modelling and simulation
Online exam August 23, 2021

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This exam contains 11 pages (including this cover page) and 5 problems.

A brief summary of instructions (detailed instructions available at Canvas):

- You must be logged into Zoom during the entire exam, with video on and yourself clearly visible against a neutral background. The microphone shall be muted, and the audio can be switched off, unless you are asked to switch it on. It is prohibited to use any kind of headphones or earphones, or to record the examination with your own equipment.
- You must be alone in the room where you conduct the exam (unless you have informed us prior to the exam).
- Please check "announcements" on the Canvas page now and then for messages from the examiner.
- If you want to contact the proctor, write "Contact" in the Zoom chat to "everyone". If you have a question for the examiner, write "Question for examiner".
- If you need to go to the bathroom, write "Bathroom" and "Bathroom return", respectively. Keep bathroom breaks as brief as possible!
- It is not allowed to cooperate or receive help from another person, or to communicate orally or in writing with anyone except the proctor and the examiner! If this is observed, it will be reported.

Solutions and submissions:

- Solutions are written by hand on paper, exactly as in a regular exam hall. Label each sheet of paper with your name, problem number and page number.
- At **12:30** at the latest, start scanning your solutions. Write "scanning solutions" in the chat. Compile your scanned solutions into one document, e.g. using Word, and save it as **one** pdf file.
- Check that the file is readable and then submit it via the Assignment in Canvas before **13:00**, which is a hard deadline! Write "Submitted in Canvas" in the chat.
- When the proctor has checked you off the list and tagged your name with `##DONE##`, you may leave the Zoom meeting by selecting "Leave breakout room" and then "Leave meeting".
- If you experience any problems with the submission in Canvas, then send the file via email to the examiner (bo.egardt@chalmers.se), before the deadline at 13:00.

Guidelines:

- Organize your work in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering may receive less credit.
- Mysterious or unsupported answers will not receive credit, but an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- None of the proposed questions require extremely long computations. If you get caught in endless algebra, you have probably missed the simple way of doing it.
- The nominal grade limits are 20 (3), 27 (4) and 34 (5).

Problem	Points	Score
1	8	
2	9	
3	10	
4	8	
5	5	
Total:	40	

GOOD LUCK !!

1. (a) (2 points) Transform the system

$$(\dddot{y})^3 + 3y = u$$

into state-space form.

(b) (2 points) The system

$$\begin{aligned}\dot{x}_1 &= -x_1 \\ \dot{x}_2 &= x_1 - 800x_2 + 400u\end{aligned}$$

should be simulated with initial condition $x(0) = [1\ 0]^\top$ when driven by a high-frequency input u. Determine a simplified system, which approximates the original system's behaviour during an initial period of simulation.

(c) (2 points) Consider the following ARMA model:

$$y(t) + 0.7y(t-1) = u(t-1) + 0.5u(t-2) + e(t) + 0.2e(t-1)$$

Find the corresponding plant and noise model transfer functions G and H, and give an *explicit* (with numeric coefficients) difference equation, showing how the one-step ahead prediction of the output is computed from data.

(d) (2 points) The system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} x$$

shall be simulated using Euler's backward method $x_{k+1} = x_k + \Delta t \cdot f(x_{k+1})$. For what values of $\Delta t > 0$ is the method stable?

Solution:

(a) By choosing the state variables $x_1 = y$ and $x_2 = \dot{y}$, the following model is obtained:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= (u - 3x_1)^{\frac{1}{3}}\end{aligned}$$

(b) The eigenvalues of the system are $\lambda_1 = -1$ and $\lambda_2 = -800$, indicating that the system is stiff. Since we are interested in the fast dynamics, we can approximate the slow state x_1 with its initial value, i.e. $x_1 \approx 1$. The simplified (fast) dynamics is then described by

$$\dot{x}_2 = -800x_2 + 400u + 1$$

(c) The plant and noise model transfer functions are

$$G(q) = \frac{B(q)}{A(q)} = \frac{q^{-1} + 0.5q^{-2}}{1 + 0.7q^{-1}} \qquad H(q) = \frac{C(q)}{A(q)} = \frac{1 + 0.2q^{-1}}{1 + 0.7q^{-1}}$$

and the one-step ahead prediction is calculated from

$$C(q)\hat{y}(t|t-1) = B(q)u(t) + \big(C(q) - A(q)\big)y(t),$$

giving the explicit expression

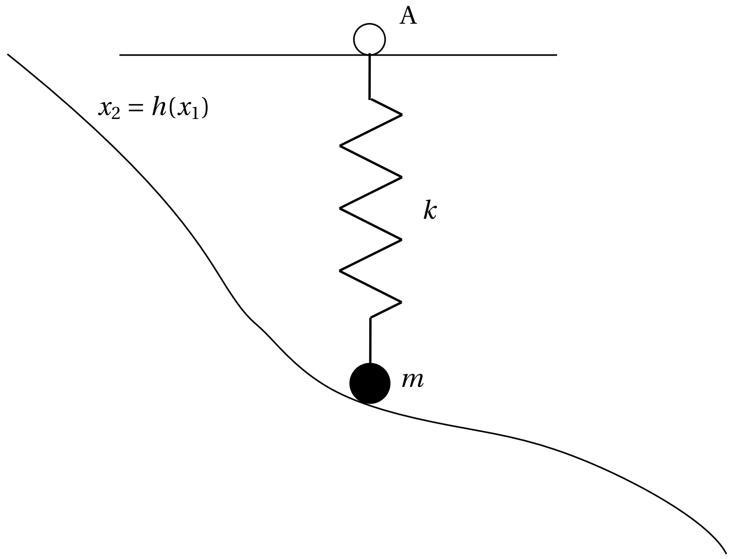
$$\hat{y}(t|t-1) = -0.2\hat{y}(t-1|t-2) + u(t-1) + 0.5u(t-2) - 0.5y(t-1)$$

(d) Applying Euler backward to the system gives the recursion

$$\begin{bmatrix} 1+\Delta t & 0 \\ 0 & 1+4\Delta t \end{bmatrix} x_{k+1} = x_k \quad \Rightarrow \quad x_{k+1} = \begin{bmatrix} \frac{1}{1+\Delta t} & 0 \\ 0 & \frac{1}{1+4\Delta t} \end{bmatrix} x_k$$

which is stable for all $\Delta t > 0$.

2. Consider the mechanical system depicted below. A ball with (point-like) mass m has the position (x_1, x_2), where x_1 is the horizontal and x_2 the vertical coordinate. The ball is gliding without friction along a rail that is described by the relation $x_2 = h(x_1)$. Further, the ball is attached to one end of a spring, having the spring constant k. The other end of the spring (A) is gliding without friction along a horizontal rail, so that the spring is always vertical.
The forces acting on the ball are thus the spring force (assuming the neutral position of the force corresponds to $x_2 = 0$), gravity g, and the normal force from the rail.



(a) (2 points) Determine the Lagrange function for the system.
(b) (2 points) Derive a dynamic model of the system in DAE form, and verify that only 2 independent initial conditions can be specified for the model.
(c) (2 points) How can you check in a simulation that there is not a tendency of the ball losing contact with the rail?
(d) (3 points) Derive a standard state-space (ODE) model of the system.
Hint: Use the constraint equation to make substitutions.

Solution:

(a) Using $\mathbf{q} = \mathbf{p} = (x_1, x_2)$, the kinetic and potential energies of the system can be written:

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2), \qquad V = mgx_2 + \frac{1}{2}kx_2^2 \tag{1}$$

With the constraint $c(\mathbf{q}) = x_2 - h(x_1) = 0$, the Lagrange function then reads as:

$$\mathcal{L} = T - V - zc = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - mgx_2 - \frac{1}{2}kx_2^2 - z(x_2 - h(x_1)) \tag{2}$$

(b) The dynamics are constructed using:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}^\top = m\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}, \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{q}}^\top = -mg\begin{bmatrix} 0 \\ 1 \end{bmatrix} - kx_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} - z\begin{bmatrix} -h'(x_1) \\ 1 \end{bmatrix} \tag{3}$$

Adding the rail constraint, the model then follows from Euler-Lagrange's equation:

$$m\begin{bmatrix}\ddot{x}_1\\ \ddot{x}_2\end{bmatrix} + mg\begin{bmatrix}0\\1\end{bmatrix} + z\begin{bmatrix}-h'(x_1)\\1\end{bmatrix} + kx_2\begin{bmatrix}0\\1\end{bmatrix} = 0 \tag{4a}$$

$$x_2 - h(x_1) = 0 \tag{4b}$$

(The DAE model can be transformed into standard semi-explicit form, but that is not required in the problem formulation.)
The two 2nd order differential equations need 4 initial conditions, namely $x_1(0), x_2(0), \dot{x}_1(0)$, and $\dot{x}_2(0)$), but the constraint and its time derivative restricts these:

$$x_2(0) = h(x_1(0)) \tag{5}$$

$$\dot{x}_2(0) = h'(x_1(0))\dot{x}_1(0), \tag{6}$$

implying that only two independent initial conditions can be given.

(c) Looking in the vertical (x_2) direction, the rail can only exert a force upwards, i.e. in the opposite direction compared to gravity. From (4a), this can be seen to correspond to $z < 1$, which is the condition that need to hold during the entire simulation.

(d) Differentiating the constraint equation gives

$$\dot{x}_2 = h'(x_1)\dot{x}_1 \tag{7a}$$

$$\ddot{x}_2 = h''(x_1)\dot{x}_1^2 + h'(x_1)\ddot{x}_1 \tag{7b}$$

Combining this with the 2nd row of (4a), we can solve for z:

$$z = -m\ddot{x}_2 - mg - kx_2 = -m\big(h''(x_1)\dot{x}_1^2 + h'(x_1)\ddot{x}_1\big) - mg - kx_2 \tag{8}$$

Inserting this expression into the first row of (4a) now gives a differential equation for x_1:

$$m\big(1 + h'(x_1)^2\big)\ddot{x}_1 + mh'(x_1)h''(x_1)\dot{x}_1^2 + (mg + kx_2)h'(x_1) = 0. \tag{9}$$

Using the state-variables x_1 and $v_1 = \dot{x}_1$, the following state-space model is finally obtained:

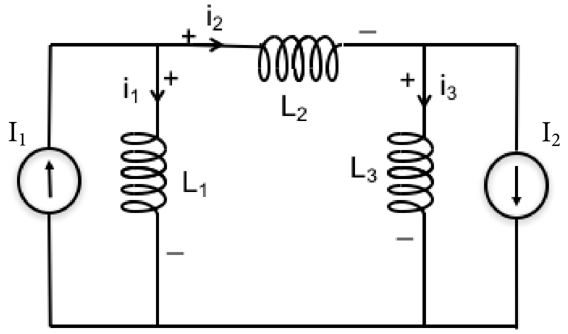
$$\dot{x}_1 = v_1 \tag{10a}$$

$$\dot{v}_1 = -\frac{1}{1 + h'(x_1)^2}\big(h'(x_1)h''(x_1)v_1^2 + (g + \frac{k}{m}x_2)h'(x_1)\big) \tag{10b}$$

By adding an "output equation" for x_2, the model completely describes the system:

$$x_2 = h(x_1). \tag{11}$$

3. Consider the electrical circuit depicted below, where two current sources $I_1(t)$ and $I_2(t)$ are driving a combination of three inductors L_1, L_2, L_3.



(a) (4 points) Determine a DAE for the circuit, using as variables the inductor currents and voltages, and with I_1, I_2 as inputs.

(b) (4 points) What is the index of the DAE?

(c) (2 points) Would the index of the DAE be reduced by replacing the current source I_1 with a voltage source U_1?

Solution:

(a) Introducing the inductor voltages u_1, u_2, u_3 and using Kirchhoff's laws along with the inductors' constitutive relations gives the following DAE in terms of the variables i_1, i_2, i_3, u_2, u_3:

$$\begin{aligned} L_1\frac{di_1}{dt} &= u_1 = u_2 + u_3 \\ L_2\frac{di_2}{dt} &= u_2 \\ L_3\frac{di_3}{dt} &= u_3 \\ I_1 &= i_1 + i_2 \\ i_2 &= i_3 + I_2 \end{aligned}$$

(b) Using $x = \begin{bmatrix} i_1 & i_2 & i_3 & u_2 & u_3 \end{bmatrix}^\top$ and $I = \begin{bmatrix} I_1 & I_2 \end{bmatrix}^\top$, we can write the DAE as

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} I$$

To determine the index, differentiate the algebraic equations to get

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{I}$$

Since the first matrix is still singular, we need to proceed. Combining the equations, we get the new DAE

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \frac{1}{L_1}+\frac{1}{L_2} & \frac{1}{L_1} \\ 0 & 0 & 0 & \frac{1}{L_2} & -\frac{1}{L_3} \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{I}$$

After once again differentiating the two algebraic equations, we get

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{L_1}+\frac{1}{L_2} & \frac{1}{L_1} \\ 0 & 0 & 0 & \frac{1}{L_2} & -\frac{1}{L_3} \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \ddot{I}.$$

It is easy to verify that the first matrix is now non-singular, i.e. the model is an ODE. Hence, the index is 2.

(c) Keeping the variables from the previous case, the new model is obtained by replacing the algebraic equation $I_1 = i_1 + i_2$ with $u_2 + u_3 = U_2$:

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} + \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U_1 \\ I_2 \end{bmatrix}$$

Now, differentiating the algebraic equations results in a coefficient matrix for $\dot{x}$ looking as

$$\begin{bmatrix} L_1 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 \\ 0 & 0 & L_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}$$

which is singular (as seen from its block diagonal structure with a singular 2x2 matrix in the lower right corner). Hence, the DAE model has at least index 2, meaning that the index is not reduced by the change.

4. We want to estimate the parameters $\theta = (b_1, b_2)$ of an FIR model with predictor

$$\hat{y}(t|t-1) = b_1 u(t-1) + b_2 u(t-2),$$

by minimizing the sum of the prediction errors squared. It is assumed that data is generated by the true system

$$y(t) = 0.5u(t-1) + 0.3u(t-2) + e(t),$$

where $e(t)$ is white noise with variance $\sigma_e^2 = 2$.

(a) (4 points) Assume the input $u(t)$ is a white noise sequence, uncorrelated with $e(t)$ (i.e. $\mathbb{E}[u(t)e(s)] = 0, \forall t, s$) and with variance $\sigma_u^2 = 3$. Compute the asymptotic (when the number of data N tends to infinity) estimate of θ as well as the asymptotic variance of the estimates.

(b) (4 points) Consider the same problem as above, but with an input u, still uncorrelated with e, having the covariance function

$$R_u(\tau) = \begin{cases} 1, & \tau = 0 \\ 1, & |\tau| = 1 \\ 0, & |\tau| > 1 \end{cases}$$

Which estimates of θ are obtained asymptotically in this case?

Solution:

(a) Find an expression for $V(\theta) = \mathbb{E}\varepsilon^2(t, \theta)$:

$$\begin{aligned} V(\theta) &= \mathbb{E}\left[(y(t) - \hat{y}(t|t-1))^2\right] \\ &= \mathbb{E}\left[((0.5 - b_1)u(t-1) + (0.3 - b_2)u(t-2) + e(t))^2\right] \\ &= \sigma_u^2\big((0.5 - b_1)^2 + (0.3 - b_2)^2\big) + \sigma_e^2, \end{aligned}$$

since all other terms disappear, due to the assumptions that $u(\cdot)$ is white noise and $u(\cdot)$ and $e(\cdot)$ are uncorrelated.
Clearly, the minimum of $V(\theta)$ corresponds to $b_1 = 0.5$ and $b_2 = 0.3$, which means that the two parameters are correctly estimated asymptotically.

Since the estimates are consistent, we can approximately evaluate the covariance of the estimates using the formula

$$\mathbb{E}\big[(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)^\top\big] \approx \frac{\sigma_e^2}{N} R^{-1},$$

with

$$R = \mathbb{E}[\varphi(t)\varphi^\top(t)] = \mathbb{E}\begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix}\begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix}^\top = \sigma_u^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

giving

$$\mathbb{E}\big[(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)^\top\big] \approx \frac{1}{N} \cdot \frac{\sigma_e^2}{\sigma_u^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \frac{2}{3N} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) A similar computation as above, but now observing that u is not any longer white noise, gives

$$\begin{aligned}
V(\theta) &= \mathbb{E}\left[(y(t) - \hat{y}(t|t-1))^2\right] \\
&= \mathbb{E}\left[((0.5 - b_1)u(t-1) + (0.3 - b_2)u(t-2) + e(t))^2\right] \\
&= \big(R_u(0)(0.5 - b_1)^2 + (0.3 - b_2)^2\big) + 2R_u(1)(0.5 - b_1)(0.3 - b_2) + \sigma_e^2 \\
&= (0.5 - b_1)^2 + (0.3 - b_2)^2 + 2(0.5 - b_1)(0.3 - b_2) + 2 \\
&= \big(0.8 - (b_1 + b_2)\big)^2 + 2
\end{aligned}$$

It is seen that any b_1, b_2, satisfying $b_1 + b_2 = 0.8$ minimizes $V(\theta)$, meaning that the parameters can not both be correctly estimated in this case; only the static gain $b_1 + b_2$ is correctly estimated.

5. (5 points) Consider a Runge-Kutta scheme for integration of an ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, defined by the following Butcher array:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

(a) Is the RK scheme explicit or implicit? How many stages are there?

(b) Write the equations describing an update of the solution sequence $\{x_k\}$.

(c) Determine the stability function.

(d) Is the scheme A-stable?

Solution:

(a) The RK scheme is implicit and has 2 stages.

(b) The RK equations are

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}(t_k)\right) \\ \mathbf{K}_2 &= \mathbf{f}\big(\mathbf{x}_k + \frac{\Delta t}{2}(\mathbf{K}_1 + \mathbf{K}_2), \mathbf{u}(t_k + \Delta t)\big) \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \frac{\Delta t}{2}(\mathbf{K}_1 + \mathbf{K}_2) \end{aligned}$$

(c) Denoting the Butcher array as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

the stability function is given by $R(\mu) = 1 + \mu b^T (I - \mu A)^{-1}\mathbf{1}$, where $\mu = \lambda\Delta t$ and $\mathbf{1}$ is a column vector with all entries equal to 1. Thus:

$$R(\mu) = 1 + \mu \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mu/2 & 1 - \mu/2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1 + \mu/2}{1 - \mu/2}$$

(d) For μ in the left half-plane, let $\mu = \alpha + i\beta$ with $\alpha < 0$, giving

$$|R(\mu)|^2 = \frac{|1 + \mu/2|^2}{|1 - \mu/2|^2} = \frac{(1 + \alpha/2)^2 + \beta^2}{(1 - \alpha/2)^2 + \beta^2} = \frac{1 + \alpha^2/4 + \beta^2 + \alpha}{1 + \alpha^2/4 + \beta^2 - \alpha} < 1.$$

Hence, $|R(\mu)| \leq 1$ for all μ in the left half-plane, i.e. the scheme is A-stable.

THE END