ESS101 Modelling and simulation
Online exam January 7, 2021

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This exam contains 10 pages (including this cover page) and 5 problems.

A brief summary of instructions (detailed instructions available at Canvas):

- You must be logged into Zoom during the entire exam, with video on and yourself clearly visible against a neutral background. The microphone shall be muted, and the audio can be switched off, unless you are asked to switch it on. It is prohibited to use any kind of headphones or earphones, or to record the examination with your own equipment.
- You must be alone in the room where you conduct the exam (unless you have informed us prior to the exam).
- Please check "announcements" on the Canvas page now and then for messages from the examiner.
- If you want to contact the proctor, write "Contact" in the Zoom chat to "everyone". If you have a question for the examiner, write "Question for examiner".
- If you need to go to the bathroom, write "Bathroom" and "Bathroom return", respectively. Keep bathroom breaks as brief as possible!
- It is not allowed to cooperate or receive help from another person, or to communicate orally or in writing with anyone except the proctor and the examiner! If this is observed, it will be reported.

Solutions and submissions:

- Solutions are written by hand on paper, exactly as in a regular exam hall. Label each sheet of paper with your name, problem number and page number.
- At **18:00** at the latest, start scanning your solutions. Write "scanning solutions" in the chat. Compile your scanned solutions into one document, e.g. using Word, and save it as **one** pdf file.
- Check that the file is readable and then submit it via the Assignment in Canvas before **18:30**, which is a hard deadline! Write "Submitted in Canvas" in the chat.
- When the proctor has checked you off the list and tagged your name with `##DONE##`, you may leave the Zoom meeting by selecting "Leave breakout room" and then "Leave meeting".
- If you experience any problems with the submission in Canvas, then send the file via email to the examiner (bo.egardt@chalmers.se), before the deadline at 18:30.

Guidelines:

- Organize your work in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering may receive less credit.
- Mysterious or unsupported answers will not receive credit, but an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- None of the proposed questions require extremely long computations. If you get caught in endless algebra, you have probably missed the simple way of doing it.
- The nominal grade limits are 20 (3), 27 (4) and 34 (5).

Problem	Points	Score
1	8	
2	8	
3	10	
4	10	
5	4	
Total:	40	

GOOD LUCK !!

1. (a) (2 points) Transform the system

$$\ddot{y} + \dot{y}^2 y + y^3 = (1 + y^2)u$$

into state-space form.

(b) (2 points) Rank the following systems according to their stiffness:

$$\text{(i)} \quad \dot{x} = \begin{bmatrix} 1 & -6 \\ 1 & -4 \end{bmatrix} x$$

$$\text{(ii)} \quad \dot{x} = \begin{bmatrix} 0 & 1 \\ -0.001 & -1.001 \end{bmatrix} x$$

$$\text{(iii)} \quad \dot{x} = \begin{bmatrix} -50 & 1 \\ 2450 & -51 \end{bmatrix} x$$

(c) (2 points) Four different ARX models are fitted to $N = 100$ input/output data-points. Depending on the number of parameters (n_a, n_b), different values of the optimal cost function

$$V_N(\hat{\theta}_N) = \min_{\theta} \frac{1}{N} \sum_{t=1}^{N} \varepsilon^2(t, \theta)$$

are obtained, as shown in the table below.

n_a	n_b	$V_N(\hat{\theta}_N)$
2	1	1.13
3	3	0.95
4	4	0.80
6	5	0.79

Using Akaike's final prediction error criterion, determine which model structure to use.

(d) (2 points) The system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix} x$$

shall be simulated using Euler's method $x_{k+1} = x_k + \Delta t \cdot f(x_k)$. For what values of $\Delta t > 0$ is the method stable?

Solution:

(a) By choosing the state variables $x_1 = y$ and $x_2 = \dot{y}$, the following model is obtained:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 x_2^2 - x_1^3 + (1 + x_1^2)u$$

(b) The stiffness is indicated by the relation between time constants or eigenvalues. For the three cases, the eigenvalues are given by $\{-1, -2\}$, $\{-0.001, -1\}$, and $\{-1, -100\}$, respectively, giving a ratio between eigenvalues of 2, 1000, and 100. Hence, (i) is the least stiff system and (ii) is the stiffest one.

(c) The FPE criterion is

$$FPE = \frac{1 + \frac{n}{N}}{1 - \frac{n}{N}} V_N(\hat{\theta}_N)$$

where the number of parameters is $n = n_a + n_b$. Evaluating the criterion for the four cases gives the following result:

n_a	n_b	n	$V_N(\hat{\theta}_N)$	FPE
2	1	3	1.13	1.20
3	3	6	0.95	1.07
4	4	8	0.80	0.94
6	5	11	0.79	0.99

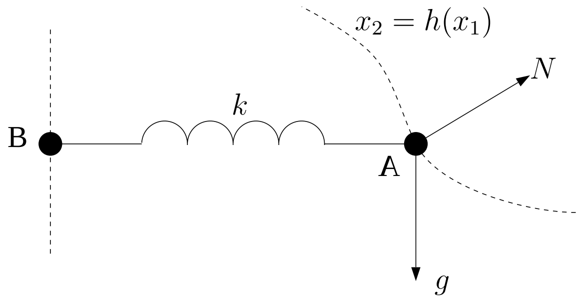
Hence, the model with $n_a = n_b = 4$ is the one to be selected.

(d) Applying Euler to the system gives the recursion

$$x_{k+1} = \begin{bmatrix} 1 - \Delta t & 0 \\ 0 & 1 - 4\Delta t \end{bmatrix} x_k,$$

which is stable for $0 < \Delta t < 0.5$.

2. Consider the mechanical system depicted below. The ball A with mass m has the position (x_1, x_2), where x_1 is the horizontal and x_2 the vertical coordinate. The ball is gliding without friction along a rail that is described by the relation $x_2 = h(x_1)$. Further, the ball A is attached to one end of a spring, having the spring constant k. The other end of the spring (B) is gliding without friction along a vertical rail, so that the spring is always horizontal.
The forces acting on A are thus the spring force (assuming the neutral position of the force corresponds to $x_1 = 0$), gravity g, and the normal force $N = (N_1, N_2)$ from the rail.



(a) (2 points) Determine the Lagrange function for the system.

(b) (2 points) Derive a dynamic model of the system in DAE form, and verify that only 2 independent initial conditions can be specified for the model.

(c) (4 points) Derive a standard state-space (ODE) model of the system.
Hint: Use the constraint equation to make substitutions.

Solution:

(a) Using $\mathbf{q} = \mathbf{p} = (x_1, x_2)$, the kinetic and potential energies of the system can be written:

$$T = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2), \qquad V = mgx_2 + \frac{1}{2}kx_1^2 \tag{1}$$

With the constraint $c(\mathbf{q}) = x_2 - h(x_1) = 0$, the Lagrange function then reads as:

$$\mathcal{L} = T - V - zc = \frac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2) - mgx_2 - \frac{1}{2}kx_1^2 - z(x_2 - h(x_1)) \tag{2}$$

(b) The dynamics are constructed using:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}}^\top = m\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix}, \qquad \frac{\partial \mathcal{L}}{\partial \mathbf{q}}^\top = -mg\begin{bmatrix} 0 \\ 1 \end{bmatrix} - kx_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} - z\begin{bmatrix} -h'(x_1) \\ 1 \end{bmatrix} \tag{3}$$

Adding the rail constraint, the model then follows from Euler-Lagrange's equation:

$$m\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + mg\begin{bmatrix} 0 \\ 1 \end{bmatrix} + z\begin{bmatrix} -h'(x_1) \\ 1 \end{bmatrix} + kx_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0 \tag{4a}$$

$$x_2 - h(x_1) = 0 \tag{4b}$$

(The DAE model can be transformed into standard semi-explicit form, but that is not required in the problem formulation.)
The two 2nd order differential equations need 4 initial conditions, namely

$x_1(0), x_2(0), \dot{x}_1(0)$, and $\dot{x}_2(0)$), but the constraint and its time derivative restricts these:

$$x_2(0) = h(x_1(0)) \tag{5}$$
$$\dot{x}_2(0) = h'(x_1(0))\dot{x}_1(0), \tag{6}$$

implying that only two independent initial conditions can be given.

(c) Differentiating the constraint equation gives

$$\dot{x}_2 = h'(x_1)\dot{x}_1 \tag{7a}$$
$$\ddot{x}_2 = h''(x_1)\dot{x}_1^2 + h'(x_1)\ddot{x}_1 \tag{7b}$$

Combining this with the 2nd row of (4a), we can solve for z:

$$z = -m\ddot{x}_2 - mg = -m\big(h''(x_1)\dot{x}_1^2 + h'(x_1)\ddot{x}_1\big) - mg \tag{8}$$

Inserting this expression into the first row of (4a) now gives a differential equation for x_1:

$$m\big(1 + h'(x_1)^2\big)\ddot{x}_1 + mh'(x_1)h''(x_1)\dot{x}_1^2 + mgh'(x_1) + kx_1 = 0. \tag{9}$$

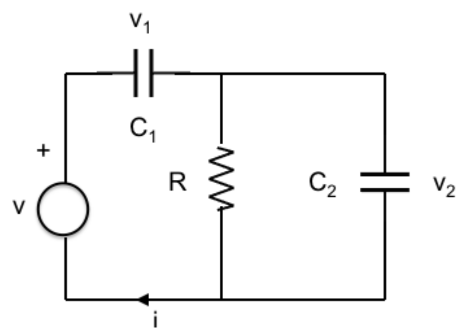
Using the state-variables x_1 and $v_1 = \dot{x}_1$, the following state-space model is finally obtained:

$$\dot{x}_1 = v_1 \tag{10a}$$
$$\dot{v}_1 = -\frac{1}{1 + h'(x_1)^2}\big(h'(x_1)h''(x_1)v_1^2 + gh'(x_1) + \frac{k}{m}x_1\big) \tag{10b}$$

By adding an "output equation" for x_2, the model completely describes the system:

$$x_2 = h(x_1). \tag{11}$$

3. Consider the electrical circuit depicted below, with a voltage source $v(t)$ driving a combination of two capacitors C_1, C_2 and a resistor R. The voltages over the capacitors are $v_1(t), v_2(t)$, and the total current is $i(t)$.



(a) (4 points) Determine a DAE for the circuit, expressed in the variables v_1, v_2, and i, and with v as the input.

(b) (3 points) What is the index of the DAE?

(c) (3 points) Let Q_1, Q_2 be the charges of the capacitors, and let $Q = Q_1 - Q_2$. Show that the model can be written in the form

$$\begin{aligned}\dot{Q} &= AQ + Bv \\ i &= CQ + D_0 v + \ldots + D_{k-1} v^{k-1},\end{aligned}$$

for some $A, B, C, D_0, \ldots, D_{k-1}$, where k is the index of the system, and v^j is the jth derivative of v.

Solution:

(a) Using Kirchhoff's laws and the components' constitutive relations gives the DAE

$$\begin{aligned}C_1 \frac{dv_1}{dt} &= i \\ C_2 \frac{dv_2}{dt} &= i - \frac{v_2}{R} \\ v &= v_1 + v_2\end{aligned}$$

(b) To determine the index, differentiate the algebraic equation to get

$$\dot{v} = \dot{v}_1 + \dot{v}_2 = \frac{1}{C_1} i + \frac{1}{C_2} i - \frac{1}{RC_2} v_2 = (\frac{1}{C_1} + \frac{1}{C_2}) i - \frac{1}{RC_2} v_2$$

Combining this with the two differential equations gives the model equations

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{E_1} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{R} & 1 \\ 0 & \frac{1}{RC_2} & -(\frac{1}{C_1} + \frac{1}{C_2}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{v},$$

where the matrix E_1 is singular, i.e. the model is still a DAE. One more differentiation gives

$$\underbrace{\begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & -\frac{1}{RC_2} & (\frac{1}{C_1} + \frac{1}{C_2}) \end{bmatrix}}_{E_2} \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \\ \dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{R} & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ddot{v},$$

where the matrix E_2 is now non-singular, i.e. the model is an ODE. Hence, the index is 2.

(c) We have $Q = Q_1 - Q_2 = C_1 v_1 - C_2 v_2$, so that

$$\dot{Q} = C_1 \dot{v}_1 - C_2 \dot{v}_2 = \frac{v_2}{R}$$

To get an expression for v_2, use the voltage relation to get

$$Q = C_1 v_1 - C_2 v_2 = C_1 v - (C_1 + C_2) v_2 \quad \Rightarrow \quad v_2 = \frac{C_1 v - Q}{C_1 + C_2},$$

which now yields

$$\dot{Q} = \frac{C_1 v - Q}{R(C_1 + C_2)} = \underbrace{-\frac{1}{R(C_1 + C_2)}}_{A} Q + \underbrace{\frac{C_1}{R(C_1 + C_2)}}_{B} v.$$

The output equation can be obtained as follows:

$$\begin{aligned} i = C_1 \dot{v}_1 = C_1(\dot{v} - \dot{v}_2) &= C_1\Big(\dot{v} - \frac{C_1 \dot{v} - \dot{Q}}{C_1 + C_2}\Big) \\ &= \frac{C_1 C_2}{C_1 + C_2}\dot{v} + \frac{C_1}{C_1 + C_2}\Big(-\frac{1}{R(C_1 + C_2)}Q + \frac{C_1}{R(C_1 + C_2)}v\Big) \\ &= \underbrace{-\frac{C_1}{R(C_1 + C_2)^2}}_{C} Q + \underbrace{\frac{C_1^2}{R(C_1 + C_2)^2}}_{D_0} v + \underbrace{\frac{C_1 C_2}{C_1 + C_2}}_{D_1} \dot{v} \end{aligned}$$

Remark: in this model form, it is clearly seen that only one independent initial condition can be specified.

4. We want to estimate the parameters $\theta = (b_1, b_2)$ of an FIR model with predictor

$$\hat{y}(t|t-1) = b_1 u(t-1) + b_2 u(t-2).$$

(a) (4 points) Assume the data is generated by the "true" system

$$y(t) = 0.2u(t-1) + 0.4u(t-2) + 0.5u(t-3) + e(t),$$

where the input $u(t)$ and the noise $e(t)$ are uncorrelated (i.e $\mathbb{E}[u(t)e(s)] = 0, \forall t, s$) and both are white noise sequences. Compute the asymptotic (when the number of data N tends to infinity) estimate of θ.

(b) (4 points) With the same model, assume instead that the true system is given by

$$y(t) = 0.2u(t-1) + 0.4u(t-2) + e(t),$$

and that the white noise $e(t)$ has variance 1. In this case, the input u, still uncorrelated with e, has the covariance function

$$R_u(\tau) = \begin{cases} 1, & \tau = 0 \\ 0.5, & |\tau| = 1 \\ 0, & |\tau| > 1 \end{cases}$$

Compute the asymptotic estimate of θ.

(c) (2 points) What is the approximate covariance of the estimates for finite N in (b) above?

Solution:

(a) Find an expression for $V(\theta) = \mathbb{E}\varepsilon^2(t,\theta)$:

$$\begin{aligned} V(\theta) &= \mathbb{E}\left[(y(t) - \hat{y}(t|t-1))^2\right] \\ &= \mathbb{E}\left[((0.2 - b_1)u(t-1) + (0.4 - b_2)u(t-2) + 0.5u(t-3) + e(t))^2\right] \\ &= \left((0.2 - b_1)^2 + (0.4 - b_2)^2 + 0.5^2\right)R_u(0) + R_e(0), \end{aligned}$$

since all other terms disappear, due to the assumptions that $u(\cdot)$ is white noise and $u(\cdot)$ and $e(\cdot)$ are uncorrelated.
Clearly, the minimum of $V(\theta)$ corresponds to $b_1 = 0.2$ and $b_2 = 0.4$, which means that the two parameters are correctly estimated asymptotically, even though the model does not contain any b_3 parameter.

(b) A similar computation as above, but now observing that u is not any longer white noise, gives

$$\begin{aligned} V(\theta) &= \mathbb{E}\left[(y(t) - \hat{y}(t|t-1))^2\right] \\ &= \mathbb{E}\left[((0.2 - b_1)u(t-1) + (0.4 - b_2)u(t-2) + e(t))^2\right] \\ &= \left((0.2 - b_1)^2 + (0.4 - b_2)^2\right)R_u(0) + 2(0.2 - b_1)(0.4 - b_2)R_u(1) + R_e(0) \\ &= (0.2 - b_1)^2 + (0.4 - b_2)^2 + (0.2 - b_1)(0.4 - b_2) + 1 \end{aligned}$$

To find the minimum, we evaluate the gradient:

$$\nabla V(\theta) = \begin{bmatrix} -2(0.2 - b_1) - (0.4 - b_2) \\ -2(0.4 - b_2) - (0.2 - b_1) \end{bmatrix} = \begin{bmatrix} 2b_1 + b_2 - 0.8 \\ b_1 + 2b_2 - 1 \end{bmatrix} = 0,$$

which gives the solution $b_1 = 0.2, b_2 = 0.4$, showing that the estimates are consistent also in this case.

(c) Since the estimates are consistent, we can approximately evaluate the covariance of the estimates using the formula

$$\mathbb{E}\left[(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)^\top\right] \approx \frac{R_e(0)}{N} R^{-1},$$

with

$$R = \mathbb{E}[\varphi(t)\varphi^\top(t)] = \mathbb{E}\begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix}\begin{bmatrix} u(t-1) \\ u(t-2) \end{bmatrix}^\top = \begin{bmatrix} R_u(0) & R_u(1) \\ R_u(1) & R_u(0) \end{bmatrix},$$

giving

$$\mathbb{E}\left[(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)(\hat{\boldsymbol{\theta}}_N - \boldsymbol{\theta}_0)^\top\right] \approx \frac{1}{N}\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}^{-1} = \frac{1}{0.75N}\begin{bmatrix} 1 & -0.5 \\ -0.5 & 1 \end{bmatrix}$$

5. (4 points) Consider a Runge-Kutta scheme for integration of an ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, defined by the following Butcher array:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$$

(a) Is the RK scheme explicit or implicit? How many stages are there?

(b) Write the equations describing an update of the solution sequence $\{x_k\}$.

(c) Determine the stability function.

(d) Is the scheme A-stable?

Solution:

(a) The RK scheme is explicit and has 2 stages.

(b)

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}(t_k)\right) \\ \mathbf{K}_2 &= \mathbf{f}\big(\mathbf{x}_k + \Delta t \cdot \mathbf{K}_1, \mathbf{u}(t_k + \Delta t)\big) \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \frac{\Delta t}{2}(\mathbf{K}_1 + \mathbf{K}_2) \end{aligned}$$

(c) Denoting the Butcher array as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

the stability function is given by $R(\mu) = 1 + \mu b^T(I - \mu A)^{-1}\mathbf{1}$, where $\mu = \lambda\Delta t$ and $\mathbf{1}$ is a column vector with all entries equal to 1. Thus:

$$R(\mu) = 1 + \mu \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mu & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 + \mu + \mu^2/2$$

(d) Since $|R(\mu)|$ increases for large $|\mu|$, the scheme is not A-stable (this is true for all explicit RK schemes).

THE END