ESS101 Modelling and simulation
Online exam October 28, 2020

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This exam contains 11 pages (including this cover page) and 6 problems.

A brief summary of instructions (detailed instructions available at Canvas):

- You must be logged into Zoom during the entire exam, with video on and yourself clearly visible against a neutral background. The microphone shall be muted, and the audio can be switched off, unless you are asked to switch it on. It is prohibited to use any kind of headphones or earphones, or to record the examination with your own equipment.
- You must be alone in the room where you conduct the exam (unless you have informed us prior to the exam).
- Please check ”announcements” on the Canvas page now and then for messages from the examiner.
- If you want to contact the proctor, write ”Contact” in the Zoom chat to ”everyone”. If you have a question for the examiner, write ”Question for examiner”.
- If you need to go to the bathroom, write ”Bathroom” and ”Bathroom return”, respectively. Keep bathroom breaks as brief as possible!
- It is not allowed to cooperate or receive help from another person, or to communicate orally or in writing with anyone except the proctor and the examiner! If this is observed, it will be reported.

Solutions and submissions:

- Solutions are written by hand on paper, exactly as in a regular exam hall. Label each sheet of paper with your name, problem number and page number.
- At 12:30 at the latest, start scanning your solutions. Write ”scanning solutions” in the chat. Compile your scanned solutions into one document, e.g. using Word, and save it as **one** pdf file.
- Check that the file is readable and then submit it via the Assignment in Canvas before 13:00, which is a hard deadline! Write ”Submitted in Canvas” in the chat.
- When the proctor has checked you off the list and tagged your name with `##DONE##`, you may leave the Zoom meeting by selecting “Leave breakout room” and then “Leave meeting”.
- If you experience any problems with the submission in Canvas, then send the file via email to the examiner (bo.egardt@chalmers.se), before the deadline at 13:00.

Guidelines:

- Organize your work in a reasonably neat and coherent way. Work scattered all over the page without a clear ordering may receive less credit.
- Mysterious or unsupported answers will not receive credit, but an incorrect answer supported by substantially correct calculations and explanations will receive partial credit.
- None of the proposed questions require extremely long computations. If you get caught in endless algebra, you have probably missed the simple way of doing it.
- The nominal grade limits are 20 (3), 27 (4) and 34 (5).

Problem	Points	Score
1	10	
2	10	
3	5	
4	5	
5	5	
6	5	
Total:	40	

GOOD LUCK !!

1. (a) (2 points) The system

$$\dot{x}_1(t) = -350x_1 + 70u(t)$$
$$\dot{x}_2(t) = 10x_1(t) - 3x_2(t) + u(t)$$
$$y(t) = 5x_1(t) + 4x_2(t)$$

is affected by a slowly-varying input signal $u(t)$. Propose a simpler first-order model (i.e. with only one state) giving approximately the same output signal.

(b) (2 points) Assume that the step response of the system

$$\ddot{y}_1(t) + 4\dot{y}_1(t) + 3y_1(t) = u_1(t)$$

has been simulated, and that we would now want to study the step response of another system:

$$\ddot{y}_2(t) + 8\dot{y}_2(t) + 12y_2(t) = u_2(t).$$

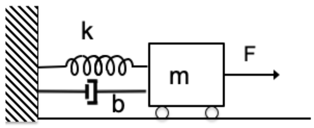
Is it possible to use the results from the first simulation by means of scaling?

(c) (2 points) Given the true system

$$y(t) = \frac{N(q)}{D(q)}u(t) + e(t),$$

where $N(q)$ and $D(q)$ are polynomials and $\{e(t)\}$ is a sequence of i.i.d. random variables, which black-box model structures give an unbiased estimator?

(d) (3 points) Give a state-space model for the simple mechanical system depicted below. Give an electrical circuit that has the same state-space model, and briefly list the correspondences between the variables and parameters in the two systems.



(e) (1 point) The optimization problem

$$\min_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y})$$

is often approached by solving the equation

$$\nabla_{\mathbf{x}} \Phi(\mathbf{x}, \mathbf{y}) = 0.$$

When is this approach *guaranteed* to work?

Solution:

(a) The system is given here again:

$$\dot{x}_1(t) = -350x_1 + 70u(t) \tag{1a}$$
$$\dot{x}_2(t) = 10x_1(t) - 3x_2(t) + u(t) \tag{1b}$$
$$y(t) = 5x_1(t) + 4x_2(t) \tag{1c}$$

It can be noticed that the dynamics of (1a) is much faster than (1b), so the differential equation can be approximated by the static equation

$$0 = -5x_1 + u(t),$$

which after insertion in (1b) gives the simpler system

$$\begin{aligned} \dot{x}_2(t) &= -3x_2(t) + 3u(t) \\ y(t) &= 4x_2(t) + u(t) \end{aligned}$$

(b) Let $z(t) = \alpha y_1(\beta t) = \alpha y_1(\tau)$. Then

$$\begin{aligned} \frac{dz(t)}{dt} &= \alpha \frac{dy_1(\tau)}{d\tau}\frac{d\tau}{dt} = \alpha\beta\frac{dy_1(\tau)}{d\tau} \\ \frac{d^2z(t)}{dt^2} &= \frac{d}{d\tau}\left(\alpha\beta\frac{dy_1(\tau)}{d\tau}\right)\frac{d\tau}{dt} = \alpha\beta^2\frac{d^2y_1(\tau)}{d\tau^2}, \end{aligned}$$

giving

$$\frac{d^2z(t)}{dt^2} + 8\frac{dz(t)}{dt} + 12z(t) = \alpha\beta^2\frac{d^2y_1(\tau)}{d\tau^2} + 8\alpha\beta\frac{dy_1(\tau)}{d\tau} + 12\alpha y_1(\tau).$$

It is seen that with the choice $\alpha = 1/4$, $\beta = 2$, the expression above equals $u_1(\tau)$, which implies that the solution $z(t) = y_2(t)$ can be obtained from the solution y_1 as $y_2(t) = y_1(2t)/4$, if the input is chosen as $u_1(\tau) = u_2(t) = u_2(\tau/2)$. In this case, time scaling of the input is not needed (since it is a step).

(c) The system is on Output Error form, so an OE model will be fine. Also an ARMAX model will work, since with $A(q) = C(q)$, there will be a cancellation and the noise model becomes similar to the one in the OE case. (In addition, the full Box-Jenkins model will work, but this has not been explicitly mentioned in the course.)

(d) With position x and velocity v, we get the model

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k/m & -b/m \end{bmatrix}\begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} F$$

An electrical circuit with R, L, and C in series, driven by a voltage source V gives a similar state-space model with capacitor charge Q and current i as state variables:

$$\begin{bmatrix} \dot{Q} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/LC & -R/L \end{bmatrix}\begin{bmatrix} Q \\ i \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} V$$

with the correspondences $F \leftrightarrow V$, $x \leftrightarrow Q$, $v \leftrightarrow i$, $L \leftrightarrow m$, $1/k \leftrightarrow C$, $b \leftrightarrow R$.

(e) The approach is guaranteed to work, when the necessary condition for a minimum is also sufficient, i.e. when the problem is convex: $\nabla^2_{\mathbf{x}}\Phi(\mathbf{x}, \mathbf{y})$ is positive definite.

2. Consider a 3D "roller-coaster" created by a mass m sliding on a surface described by the (scalar-valued) constraint equation $c(\mathbf{p}) = 0$, where $\mathbf{p} \in \mathbb{R}^3$ is the usual cartesian coordinate vector. The mass is affected by gravity and subject to a friction force given by $\mathbf{F} = -m\gamma\dot{\mathbf{p}}$.
   (a) (2 points) Determine the Lagrange function for the system.
   (b) (2 points) Derive a dynamic model of the system *in semi-explicit DAE form.* How many differential and algebraic variables does the model have?
   (c) (3 points) Perform an index-reduction to get an index-1 DAE in the *fully implicit form.*
   (d) (1 point) Show how to modify the DAE you derived in (c) to avoid drift away from the constraint surface in simulations.
   (e) (2 points) Show that the DAE you derived in (c) is well-defined ("easy to solve"), except for a case of no interest.

Solution:

(a) The kinetic and potential energies of the system can be written as ($\mathbf{q} = \mathbf{p}$):

$$T = \frac{1}{2}m\dot{\mathbf{p}}^\top\dot{\mathbf{p}}, \qquad V = mg\mathbf{e}^\top\mathbf{p} \tag{2}$$

where $\mathbf{e}^\top = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$. The Lagrange function then reads as:

$$\mathcal{L} = T - V - zc = \frac{1}{2}m\dot{\mathbf{p}}^\top\dot{\mathbf{p}} - mg\mathbf{e}^\top\mathbf{p} - zc(\mathbf{p}) \tag{3}$$

(b) The dynamics are constructed using:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}^\top = m\ddot{\mathbf{p}}, \qquad \frac{\partial\mathcal{L}}{\partial\mathbf{q}}^\top = -mg\mathbf{e} - z\frac{\partial c}{\partial\mathbf{p}}^\top \tag{4}$$

Since we work in cartesian coordinates, the generalized force attached to the friction is simply $\mathbf{Q} = \mathbf{F}$. The model then follows from Euler-Lagrange's equation:

$$m\ddot{\mathbf{p}} + mg\mathbf{e} + z\frac{\partial c}{\partial\mathbf{p}}^\top = -m\gamma\dot{\mathbf{p}} \tag{5a}$$

$$0 = c(\mathbf{p}) \tag{5b}$$

In order to get a standard semi-explicit form, define the states $\mathbf{x} = \begin{bmatrix} \mathbf{p} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{p} \\ \dot{\mathbf{p}} \end{bmatrix}$ to get

$$\dot{\mathbf{p}} = \mathbf{v} \tag{6a}$$

$$\dot{\mathbf{v}} = -\gamma\mathbf{v} - g\mathbf{e} - \frac{z}{m}\frac{\partial c}{\partial\mathbf{p}}^\top \tag{6b}$$

$$0 = c(\mathbf{p}) \tag{6c}$$

The model has 6 differential variables ($\mathbf{p}, \mathbf{v} \in \mathbb{R}^3$) and 1 algebraic variable (z).

(c) Since we have a semi-explicit DAE, the index reduction requires time-differentiation of the algebraic constraint alone. We have:

$$\dot{c} = \frac{\partial c}{\partial\mathbf{p}}\dot{\mathbf{p}} = \frac{\partial c}{\partial\mathbf{p}}\mathbf{v} = 0 \tag{7}$$

Since $\frac{\partial \dot{c}}{\partial z} = 0$, this equation does not deliver z yet. We therefore perform a second time-differentiation:

$$\ddot{c} = \frac{\partial}{\partial \mathbf{p}}\left(\frac{\partial c}{\partial \mathbf{p}}\mathbf{v}\right)\mathbf{v} + \frac{\partial c}{\partial \mathbf{p}}\dot{\mathbf{v}} = \mathbf{v}^\top \frac{\partial^2 c}{\partial \mathbf{p}^2}\mathbf{v} + \frac{\partial c}{\partial \mathbf{p}}\dot{\mathbf{v}} = 0 \tag{8}$$

We can then assemble the DAE:

$$\dot{\mathbf{p}} = \mathbf{v} \tag{9a}$$

$$m\dot{\mathbf{v}} + \frac{\partial c}{\partial \mathbf{p}}^\top z = -m\gamma\mathbf{v} - mg\mathbf{e} \tag{9b}$$

$$\frac{\partial c}{\partial \mathbf{p}}\dot{\mathbf{v}} = -\mathbf{v}^\top \frac{\partial^2 c}{\partial \mathbf{p}^2}\mathbf{v} \tag{9c}$$

or equivalently:

$$\underbrace{\begin{bmatrix} I_3 & 0 & 0 \\ 0 & mI_3 & \frac{\partial c}{\partial \mathbf{p}}^\top \\ 0 & \frac{\partial c}{\partial \mathbf{p}} & 0 \end{bmatrix}}_{:=M(\mathbf{p})} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -m\gamma\mathbf{v} - mg\mathbf{e} \\ -\mathbf{v}^\top \frac{\partial^2 c}{\partial \mathbf{p}^2}\mathbf{v} \end{bmatrix} \tag{10a}$$

(d) Drift is avoided by replacing the constraint $\ddot{c} = 0$ with $\ddot{c} + \alpha_1\dot{c} + \alpha_2 c = 0$, with $\alpha_1, \alpha_2 > 0$. Hence:

$$\begin{bmatrix} I_3 & 0 & 0 \\ 0 & mI_3 & \frac{\partial c}{\partial \mathbf{p}}^\top \\ 0 & \frac{\partial c}{\partial \mathbf{p}} & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{v}} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{v} \\ -m\gamma\mathbf{v} - mg\mathbf{e} \\ -\mathbf{v}^\top \frac{\partial^2 c}{\partial \mathbf{p}^2}\mathbf{v} - \alpha_1\frac{\partial c}{\partial \mathbf{p}}\mathbf{v} - \alpha_2 c(\mathbf{p}) \end{bmatrix} \tag{11a}$$

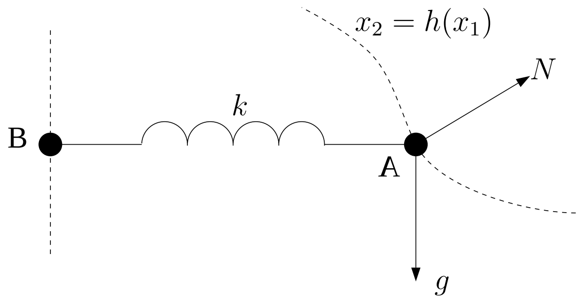
(e) The implicit DAE is well-defined if the matrix $M(\mathbf{p})$ is non-singular (implying that $\dot{\mathbf{p}}, \dot{\mathbf{v}}$, and z can be solved for, given $\mathbf{p}, \mathbf{v}$). To prove that $M(\mathbf{p})$ is non-singular, elementary row or column operations can be performed without changing the determinant. The latter gives for example

$$\begin{aligned} \det M(\mathbf{p}) = \det\left(M(\mathbf{p}) \cdot \begin{bmatrix} I_3 & 0 & 0 \\ 0 & I_3 & -\frac{1}{m}\frac{\partial c}{\partial \mathbf{p}}^\top \\ 0 & 0 & 1 \end{bmatrix}\right) &= \det \begin{bmatrix} I_3 & 0 & 0 \\ 0 & mI_3 & 0 \\ 0 & \frac{\partial c}{\partial \mathbf{p}} & -\frac{1}{m}\frac{\partial c}{\partial \mathbf{p}}\frac{\partial c}{\partial \mathbf{p}}^\top \end{bmatrix} \\ &= -\frac{\partial c}{\partial \mathbf{p}}\frac{\partial c}{\partial \mathbf{p}}^\top = -\left\|\frac{\partial c}{\partial \mathbf{p}}\right\|^2, \end{aligned} \tag{12}$$

showing that $M(\mathbf{p})$ is non-singular, as long as $\|\frac{\partial c}{\partial \mathbf{p}}\|^2$ is non-zero, which is true unless c is independent of $\mathbf{q}$ (meaning that it is actually not a constraint).

3. Consider the mechanical system depicted below. The ball A with mass m has the position (x_1, x_2), where x_1 is the horizontal and x_2 the vertical coordinate. The ball is gliding without friction along a rail that is described by the relation $x_2 = h(x_1)$. Further, the

ball A is attached to one end of a spring, having the spring constant k. The other end of the spring (B) is gliding without friction along a vertical rail, so that the spring is always horizontal.
The forces acting on A are thus the spring force (assuming the neutral position of the force corresponds to $x_1 = 0$), gravity g, and the normal force $N = (N_1, N_2)$ from the rail.



(a) (1 point) Use the relation $x_2 = h(x_1)$ to find a relation between N_1 and N_2.

(b) (1 point) Apply Newton's second law of motion to find a DAE in the variables $x_1, x_2, v_1, v_2, N_1, N_2$, describing the motion of A. Here v_1, v_2 are the horizontal and vertical velocities, respectively.

(c) (3 points) What is the index of this DAE?

Solution:

(a) N is orthogonal to the rail, implying

$$(N_1, N_2) \perp (1, \frac{dx_2}{dx_1}) = (1, h'(x_1)) \quad \Rightarrow \quad N_1 = -N_2 h'(x_1)$$

(b) The DAE becomes

$$\dot{x}_1 = v_1 \tag{13a}$$
$$\dot{x}_2 = v_2 \tag{13b}$$
$$m\dot{v}_1 = N_1 - kx_1 \tag{13c}$$
$$m\dot{v}_2 = N_2 - mg \tag{13d}$$
$$x_2 = h(x_1) \tag{13e}$$
$$N_1 = -N_2 h'(x_1) \tag{13f}$$

(c) To investigate the index, a first differentiation of the algebraic equations gives

$$\dot{x}_2 = h'(x_1)\dot{x}_1 \Leftrightarrow v_2 = h'(x_1)v_1 \tag{14a}$$
$$\dot{N}_1 = -h'(x_1)\dot{N}_2 - N_2 h''(x_1)\dot{x}_1 = -h'(x_1)\dot{N}_2 - N_2 h''(x_1)v_1, \tag{14b}$$

delivering one more differential equation. The remaining algebraic equation is differentiated once more, giving

$$\dot{v}_2 = h'(x_1)\dot{v}_1 + h''(x_1)\dot{x}_1 v_1 = h'(x_1)\dot{v}_1 + h''(x_1)v_1^2, \tag{15}$$

which, using (13c),(13d), again leads to an algebraic equation:

$$N_2 - mg = h'(x_1)(N_1 - kx_1) + mh''(x_1)v_1^2. \tag{16}$$

A third differentiation now gives

$$\begin{aligned}\dot{N}_2 &= h'(x_1)(\dot{N}_1 - kv_1) + h''(x_1)v_1(N_1 - kx_1) + 2mh''(x_1)v_1\dot{v}_1 + mv_1^2h'''(x_1)v_1 \\ &= h'(x_1)(\dot{N}_1 - kv_1) + h''(x_1)v_1(N_1 - kx_1) + 2h''(x_1)v_1(N_1 - kx_1) + mv_1^2h'''(x_1)v_1,\end{aligned} \tag{17}$$

thus finally giving an ODE by combining the original 4 differential equations with (14b) and (17). Hence, the original DAE has index 3.

4. (5 points) Consider the model structure

$$y(t) + \alpha y(t-1) = u(t-1) + e(t) + \gamma e(t-1),$$

where $e(\cdot)$ is a sequence of i.i.d. random variables with zero mean, and the coefficient for u is known to be 1.

(a) Compute the one-step ahead predictor $\hat{y}(t|t-1)$.

(b) What condition(s) need to be fulfilled for this predictor to be useful in practice?

(c) We want to fit the above model to data by minimizing the quadratic criterion

$$V_N(\theta) = \frac{1}{N}\sum_{t=1}^{N}(y(t) - \hat{y}(t|t-1))^2$$

with $\theta^T = \begin{bmatrix}\alpha & \gamma\end{bmatrix}$. How can the minimizing $\boldsymbol{\theta}$ be found? Answer brief and concise!

Solution: Using the backward shift operator q^{-1} (alternatively, z^{-1} can be used), the model can equivalently be written as

$$(1 + \alpha q^{-1})y(t) = q^{-1}u(t) + (1 + \gamma q^{-1})e(t). \tag{18}$$

(a) The derivation is given for the general case in the Lecture notes, but in this particular case it can be done simpler. Rewrite the model as

$$(1 + \gamma q^{-1})y(t) = (\gamma - \alpha)q^{-1}y(t) + q^{-1}u(t) + (1 + \gamma q^{-1})e(t), \tag{19}$$

or, equivalently,

$$y(t) = \frac{1}{1 + \gamma q^{-1}}\left[(\gamma - \alpha)q^{-1}y(t) + q^{-1}u(t)\right] + e(t) = \hat{y}(t|t-1) + e(t), \tag{20}$$

where the latter equality follows from the fact that $e(t)$ is the only part of $y(t)$ that cannot be predicted. Hence, the predictor is given by the difference equation

$$(1 + \gamma q^{-1})\hat{y}(t|t-1) = (\gamma - \alpha)y(t-1) + u(t-1) \tag{21}$$

(b) The predictor is a dynamic system with inputs u and y and needs to be stable to produce useful results. For this, we need to secure that $|\gamma| < 1$.

(c) Since $\hat{y}$ depends nonlinearly on $\boldsymbol{\theta}$, the minimization is approached by iteratively searching for a solution to the equation

$$\nabla_{\boldsymbol{\theta}} V_N(\boldsymbol{\theta}) = 0$$

5. (5 points) We want to estimate the parameters $\theta = (b_1, b_2)$ of an FIR model with predictor

$$\hat{y}(t|t-1) = b_1 u(t-1) + b_2 u(t-2).$$

Assume the data is generated by the "true" system

$$y(t) = u(t-1) + 0.7u(t-2) + e(t),$$

where both the input $u(t)$ and the noise $e(t)$ have zero mean and are assumed to be uncorrelated (i.e $\mathbb{E}[u(t)e(s)] = 0, \forall t, s$). Their covariance functions are given by

$$R_u(\tau) = \frac{1}{2^{|\tau|}}, \quad R_e(\tau) = \frac{1}{3^{|\tau|}}, \quad \forall\tau.$$

Compute the asymptotic (when the number of data tends to infinity) estimate of θ.
Hint: recall that the parameters converge to the minimum of the function $\mathbb{E}[\varepsilon^2(t,\theta)]$.

Solution: Find an expression for $V(\theta) = \mathbb{E}\varepsilon^2(t,\theta)$:

$$\begin{aligned}
V(\theta) &= \mathbb{E}\left[(y(t) - \hat{y}(t|t-1))^2\right] \\
&= \mathbb{E}\left[((1-b_1)u(t-1) + (0.7-b_2)u(t-2) + e(t))^2\right] \\
&= (1-b_1)^2 R_u(0) + (0.7-b_2)^2 R_u(0) + 2(1-b_1)(0.7-b_2)R_u(1) + R_e(0) \\
&\quad \text{(all other terms disappear, since } u(\cdot) \text{ and } e(\cdot) \text{ are uncorrelated)} \\
&= (1-b_1)^2 + (0.7-b_2)^2 + (1-b_1)(0.7-b_2) + 1
\end{aligned}$$

To find the minimum, we evaluate the gradient:

$$\nabla V(\theta) = \begin{bmatrix} -2(1-b_1) - (0.7-b_2) \\ -2(0.7-b_2) - (1-b_1) \end{bmatrix} = \begin{bmatrix} 2b_1 + b_2 - 2.7 \\ b_1 + 2b_2 - 2.4 \end{bmatrix} = 0,$$

which gives the solution $b_1 = 1, b_2 = 0.7$, showing that the estimates are consistent, even though the noise is not white; this is due to the fact that the input and the noise are uncorrelated. *Remark:* it can easily be verified that the computed extremum point is really a minimum.

6. (5 points) Consider the following Runge-Kutta equations for integration of an ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$:

$$\begin{aligned} \mathbf{K}_1 &= \mathbf{f}\left(\mathbf{x}_k, \mathbf{u}(t_k)\right) \\ \mathbf{K}_2 &= \mathbf{f}\big(\mathbf{x}_k + \frac{\Delta t}{2}(\mathbf{K}_1 + \mathbf{K}_2), \mathbf{u}(t_k + \Delta t)\big) \\ \mathbf{x}_{k+1} &= \mathbf{x}_k + \frac{\Delta t}{2}(\mathbf{K}_1 + \mathbf{K}_2) \end{aligned}$$

(a) Determine, if possible, the number of stages and the order of the scheme, and whether it is an explicit or implicit RK scheme.

(b) What is the Butcher array describing the scheme?

(c) Determine the stability function.

(d) Is the scheme A-stable?

Solution:

(a) The RK scheme is implicit and has 2 stages, but the order cannot be determined in a straightforward way.

(b) The Butcher array is given by

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1/2 & 1/2 \\ \hline & 1/2 & 1/2 \end{array}$$

(c) Denoting the Butcher array as

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

the stability function is given by $R(\mu) = 1 + \mu b^T (I - \mu A)^{-1}\mathbf{1}$, where $\mu = \lambda \Delta t$ and $\mathbf{1}$ is a column vector with all entries equal to 1. Thus:

$$R(\mu) = 1 + \mu \begin{bmatrix} 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\mu/2 & 1-\mu/2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1+\mu/2}{1-\mu/2}$$

(d) Since $|1 + \mu/2| \leq |1 - \mu/2|$ for all μ in the left half-plane, $|R(\mu)| \leq 1$ for the same μ, i.e. the scheme is A-stable.

THE END