SOLUTIONS OF THE MODELING AND SIMULATION EXAM (ES5101)

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Problem 1

(a) Sufficient conditions for convergence are 1) consistency and 2) zero-stability

Define the two polynomials

$$\rho(r) = r^2 + \alpha_1 r^0 + \alpha_0 \quad \text{and} \quad \sigma(r) = \beta_2 r^2 + \beta_1 r + \beta_0$$

The consistency condition is:

$$\rho(1) = 0 \ , \ \rho'(1) = \sigma(1) \quad \text{that is}$$

$$\begin{cases} \alpha_1 + \alpha_0 = -1 \\ 2 + \alpha_1 = \beta_2 + \beta_1 + \beta_0 \end{cases}$$

(b) The coefficients must be chosen such that

1) $|r| < 1$ for $\rho(r)$ and

2) $x(t+2h) + \alpha_1 x(t+h) + \alpha_0 x(t) - h\left[\beta_2 f(t+2h) + \beta_1 f(t+h) + \beta_0 f(t)\right] = O(h^{p+1})$ with highest possible p.

where h is the step size

Condition 1)

$$\rho(r) = 0 \Rightarrow r = \frac{-\alpha_1 \pm \sqrt{\alpha^2 - 4\alpha_0}}{2} \qquad |r| < 1, \ |r| = 1 \text{ simple}$$

Condition 2)

Expand $x(t+2h)$, $x(t+h)$, $f(t+2h)\ (= \dot{x}(t+2h))$, $f(t+h)\ (= \dot{x}(t+h))$ in Taylor series around ~~...~~ t

$$x(t+2h) = x(t) + 2h\dot{x}(t) + 2h^2\ddot{x}(t) + \frac{1}{6}8h^3\dddot{x}(t) + O(h^4)$$

$$x(t+h) = x(t) + h\dot{x}(t) + \frac{1}{2}h^2\ddot{x}(t) + \frac{1}{6}h^3\dddot{x}(t) + O(h^4)$$

$$\dot{x}(t+2h) = \dot{x}(t) + 2h\ddot{x}(t) + 2h^2\dddot{x}(t) + \frac{1}{6}8h^3x^{(4)}(t) + O(h^4)$$

$$\dot{x}(t+h) = \dot{x}(t) + h\ddot{x}(t) + \frac{1}{2}h^2\dddot{x}(t) + \frac{1}{6}h^3x^{(4)}(t) + O(h^4)$$

The condition in 2) becomes

$$\max_{\alpha_0,\alpha_1,\beta_0,\beta_1,\beta_2} p$$

s.t.

$$
\begin{aligned}
& x + \dot{x}2h + 2h^2\ddot{x} + \frac{8}{6}h^3\dddot{x} + O(h^4) + \\
& \alpha_1\left[x + \dot{x}h + \frac{1}{2}h^2\ddot{x} + \frac{1}{6}h^3\dddot{x} + O(h^4)\right] + \\
& \alpha_0 x = \\
& h\beta_2\left[\dot{x} + 2h\ddot{x} + 2h^2\dddot{x} + \frac{8}{6}h^3x^{(4)} + O(h^4)\right] + \\
& h\beta_1\left[\dot{x} + h\ddot{x} + \frac{1}{2}h^2\dddot{x} + \frac{1}{6}h^3x^{(4)} + O(h^4)\right] + \\
& h\beta_0\dot{x} + O(h^{p+1})
\end{aligned}
$$

$$\Downarrow$$

$$
\begin{aligned}
& (1+\alpha_1+\alpha_0)x + (2h + \alpha_1 h - h\beta_2 - h\beta_1 - h\beta_0)\dot{x} + \\
& + \left(2h^2 + \frac{1}{2}\alpha_1 h^2 - 2h^2\beta_2 - h^2\beta_1\right)\ddot{x} + \\
& + \left(\frac{4}{3}h^3 + \frac{1}{6}h^3\alpha_1 - 2h^3\beta_2 - \frac{1}{2}h^3\beta_1\right)\dddot{x} \pm \cancel{O(h^{p+1})} + O(h^4)
\end{aligned}
$$

$$\Updownarrow$$

$$
\begin{cases}
1+\alpha_1+\alpha_0 = 0 \\
\beta_0+\beta_1+\beta_2 = -2-\alpha_1 \\
\beta_1 + 2\beta_2 = 2 + 1/2\,\alpha_1 \\
\frac{1}{2}\beta_1 + 2\beta_2 = \frac{4}{3} + \frac{1}{6}\alpha_1
\end{cases}
$$

Hence, consistency and zero-stability conditions require that: (3)

$$\begin{cases} \left| \dfrac{-\alpha \pm \sqrt{\alpha_1^2 - 4\alpha_0}}{2} \right| < 1 \\ 1 + \alpha_1 + \alpha_0 = 0 \\ \beta_0 + \beta_1 + \beta_2 = -2 - \alpha_1 \\ \beta_1 + 2\beta_2 = 2 + \dfrac{1}{2}\alpha_1 \\ \dfrac{1}{2}\beta_1 + 2\beta_2 = \dfrac{4}{3} + \dfrac{1}{6}\alpha_1 \end{cases}$$

The first two conditions:

$$\alpha_0 = -1 - \alpha_1$$

$$\left| \frac{-\alpha_1 \pm \sqrt{\alpha_1^2 + 4 + 4\alpha_1}}{2} \right| = \left| \frac{-\alpha_1 \pm (\alpha_1 + 2)}{2} \right| = \begin{cases} |-\alpha_1 - 1| \\ 1 \end{cases}$$

Since $r = 1$ is a simple root, the zero-stability condition reduces to:

$$|-\alpha_1 - 1| < 0 \Rightarrow -2 < \alpha_1 < 0$$

By setting, $\boxed{\alpha_1 = -1} \Rightarrow \boxed{\alpha_0 = 0}$, corresponding to

$$\boxed{\beta_0 = -0.08, \ \beta_1 = D.66, \ \beta_2 = 0.42}$$

resulting into a 3rd-order method.

(c) It's enough to verify that the local error is $O(h^4)$. Set $h = 0.1$ and calculate

$$e_2 = e^{-3 \cdot 0.2} - \left(-\alpha_1 e^{-3 \cdot 0.1} \cdot x_0 - \alpha_0 \cdot x_0 + 3h\beta_1 e^{-3 \cdot 0.1} \cdot x_0 + 3\beta_0 \cdot x_0\right) / (1 + 0.3\beta_2)$$

By setting $x_0 = 1 \Rightarrow e_2 = -2.15 \cdot 10^{-4}$

Problem 2

(a) By defining the predictor as:

$$\hat{y}(t|\vartheta) = \vartheta^T \varphi(t) \quad \text{with} \quad \vartheta = \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \end{bmatrix} \text{ and}$$

$$\varphi(t) = \begin{bmatrix} \sqrt{y(t-1)} \\ e^{u(t-1)} \end{bmatrix}$$

The standard LS formula can be used

(b) $\hat{R}_{\varepsilon u}(\tau) \neq 0$ with $\tau > 0$ suggests a delay of τ / T_s steps in the model, where T_s is the sampling time

Problem 3

The DAE

$$\begin{cases} \dot{x} = Ax + Bu \\ x_1 - x_2 = u \end{cases}$$

is of the type: $\begin{cases} \dot{x} = f(x,u) \\ g(x,u) = 0 \end{cases}$, that has index 1

Hence 1 differentiation is required to obtain a set of ODEs:

$$\underbrace{\frac{\partial g}{\partial x}}_{g_x} \dot{x} + \underbrace{\frac{\partial g}{\partial u}}_{g_u} \cdot \dot{u} = 0 \quad \dot{u} = -g_u^{-1} g_x \cdot \dot{x}$$

with $g_u = -1$, $g_x = [1 \ -1]$. Hence:

$$\dot{u} = -1.5x_1 + 2x_2 - u$$

The resulting ODE is:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{u} \end{bmatrix} = \begin{bmatrix} -1.5 & 1 & 0 \\ 0 & -1 & 1 \\ -1.5 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ u \end{bmatrix}$$

with eigenvalues $\lambda_1 = -2.78, \lambda_2 = -0.72, \lambda_3 = 0$

Let's integrate the system with FE method and ~~for~~ $h = 0.5$. We obtain:

$$x(1) = \begin{bmatrix} 0.75 \\ 0.5 \\ 0.25 \end{bmatrix}, \quad x(2) = \begin{bmatrix} 0.4375 \\ 0.3750 \\ 0.0625 \end{bmatrix}$$

Clearly $x_1(t) - x_2(t) = u(t) \quad t = 1, 2$

Problem 4

1. True
2. False.
3. False
4. True
5. True