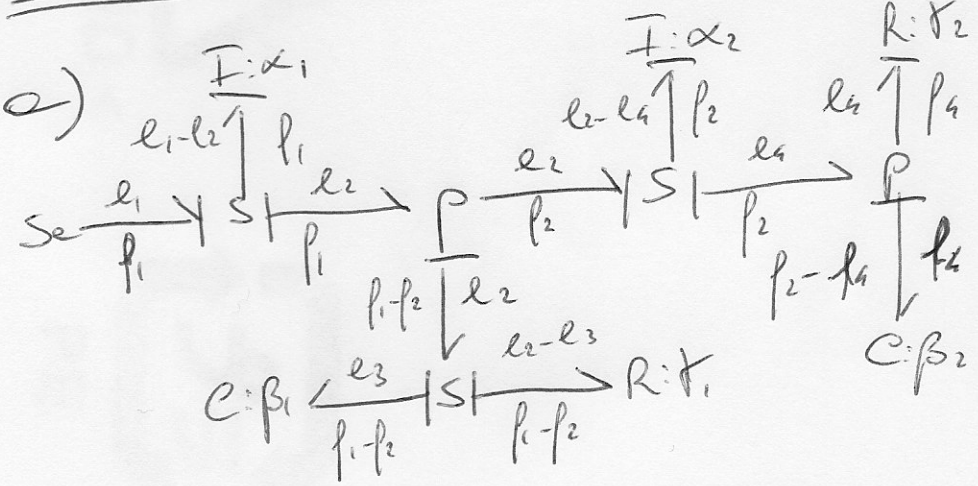
Problem 1

a)

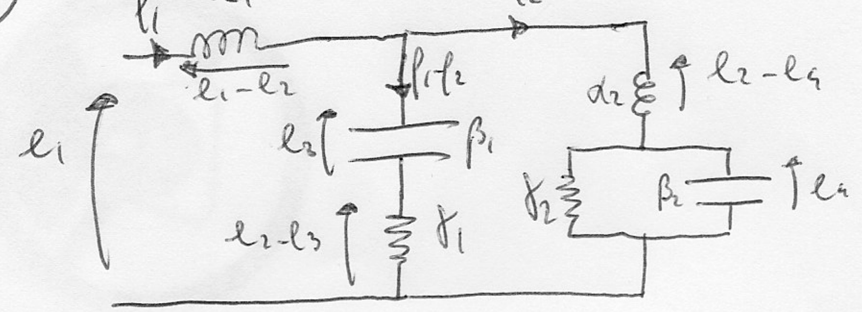


b) Define the state and the input vectors as

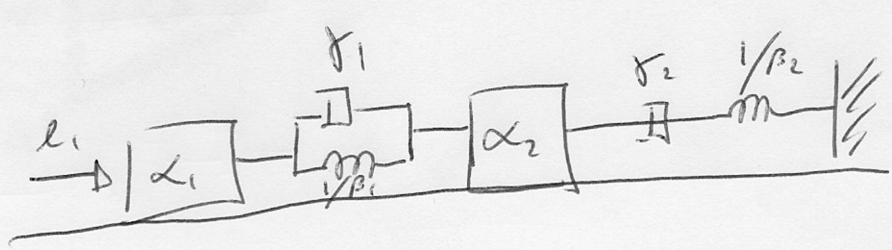
$$x = \begin{bmatrix} f_1 \\ e_3 \\ f_2 \\ e_4 \end{bmatrix} \qquad u = e_1$$

$$\begin{cases} \dot{f}_1 = \alpha_1 (e_1 - e_2) \\ \dot{e}_3 = \beta_1 (f_1 - f_2) \\ \dot{f}_2 = \alpha_2 (e_2 - e_4) \\ \dot{e}_4 = \beta_2 (f_2 - f_4) \end{cases} \qquad \begin{cases} \dot{x}_1 = -\gamma_1 \alpha_1 x_1 - \alpha_1 x_2 - \alpha_1 \gamma_1 x_3 + \alpha_1 u \\ \dot{x}_2 = \beta_1 x_1 - \beta_1 x_3 \\ \dot{x}_3 = \gamma_1 \alpha_2 x_1 + \alpha_2 x_2 - \gamma_1 \alpha_2 x_3 - \alpha_2 x_4 \\ \dot{x}_4 = \beta_2 x_3 - \dfrac{\beta_2}{\gamma_2} x_4 \end{cases}$$

c)



d)



Problem 2

We next estimate the parameters of a model ARX(1,1).

By applying the least squares formula, the optimal vector of parameters ϑ_N^* is given by:

$$\vartheta_N^* = \left(\frac{1}{N}\sum_{t=1}^{N}\begin{pmatrix} y^2(t-1) & y(t-1)u(t) \\ y(t-1)u(t) & u^2(t)\end{pmatrix}\right)^{-1}\begin{pmatrix} y(t-1)\,u(t) \\ u(t)\,y(t)\end{pmatrix}$$

As $N \to \infty$

$$\vartheta_N^* \to \begin{pmatrix} R_y(0) & R_{yu}(1) \\ R_{yu}(1) & R_u(0)\end{pmatrix}^{-1}\begin{pmatrix} R_y(1) \\ R_{uy}(0)\end{pmatrix}$$

$$R_y(0) = E[y^2(t)] = E[+0.25\,y^2(t-1) + u^2(t) + e^2(t)] =$$

$$= +0.25\,R_y(0) + 3 \Rightarrow R_y(0) = 4$$

$R_{yu}(1) = 0$, $R_u(0) = 2$, $R_{uy}(0) = 2$

$$R_y(1) = -0.5\,R_y(0) = -2$$

Hence

$$\theta_N^* \to \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} -2 \\ 2 \end{pmatrix} = \begin{pmatrix} -0.5 \\ 1 \end{pmatrix}$$

The chosen parametrization is the correct one, i.e., θ_N^* tends to the actual values as $N \to \infty$

Problem 3

a) In this case, the Backward Euler method can be used. The forward Euler method can't be used since, with the chosen integration step size, the eigenvalues of the system (-3 and -1) would ~~and~~ not lie within ~~its~~ stability region.

b) Define the system $\dot{x} = Ax + Bu$ with $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x_{k+1} = (I + hA) x_k + hB u_k$ (Forward Euler)

$x_{k+1} = (I - hA)^{-1} x_k + (I - hA)^{-1} hB u_{k+1}$ (Backward Euler)

Problem 4

1) True.
2) False.
3) False
4) True
5) True