ESS100 Modelling and simulation
Solutions to exam Mon, 22 October 2007

Exercise 1

(a) The parameters in an ARX-model can be determined in one step solving a linear equation system. For an OE-model it is necessary to use an iterative method to determine the parameters and this takes time. There is also a risk that the solution is a local minima.

(b) Index=2.

(c)

$$\Phi_y(\omega) = |G(j\omega)|^2 \Phi_u(\omega) + \Phi_e(\omega) = G(j\omega)G(-j\omega)\Phi_u(\omega) + \Phi_e(\omega)$$

$$\Phi_y(\omega) = \frac{1}{j\omega + 1}\frac{1}{-j\omega + 1}\Phi_u(\omega) + \Phi_e(\omega) = \frac{1}{\omega^2 + 1}\Phi_u(\omega) + \Phi_e(\omega)$$

Exercise 2

(a) We can write the predictor as

$$\hat{y}(t|\theta) = \theta^T \varphi(t)$$

with $\theta = [a\ b]^T$ and $\varphi = [1\ t]^T$. The least squares estimate then becomes

$$\hat{\theta} = \begin{pmatrix} \sum_{t=1}^N 1 & \sum_{t=1}^N t \\ \sum_{t=1}^N 2 & \sum_{t=1}^N t^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum_{t=1}^N y(t) \\ \sum_{t=1}^N ty(t) \end{pmatrix}$$

using the values given in the exercise gives

$$\hat{\theta} = \begin{pmatrix} 4 & 10 \\ 10 & 30 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ 10 \end{pmatrix} = \begin{pmatrix} 2.5 \\ 0.5 \end{pmatrix}$$

(b) When the noise has variance 1, the covariance matrix becomes

$$P_N = E\{(\theta_N - \theta_0)(\theta_N - \theta_0)^T\} = \frac{1}{N}\bar{R}^{-1}$$

The estimate of the covariance matrix then becomes

$$\hat{P}_N = \begin{pmatrix} \sum_{t=1}^{N} 1 & \sum_{t=1}^{N} t \\ \sum_{t=1}^{N} 2 & \sum_{t=1}^{N} t^2 \end{pmatrix}^{-1} = \begin{pmatrix} N & N(N+1)/2 \\ N(N+1)/2 & N(N+1)(2N+1)/6 \end{pmatrix}^{-1}$$

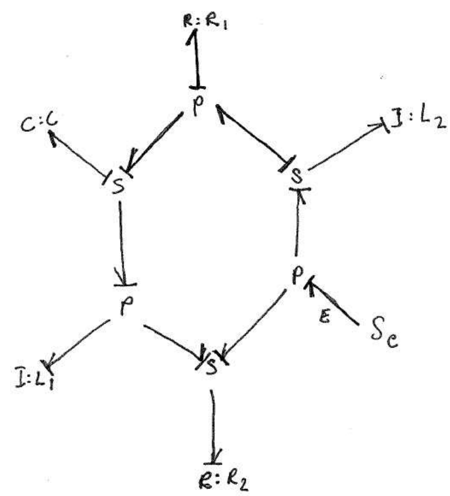
$$= \frac{12}{N^2(N+1)(N-1)} \begin{pmatrix} N(N+1)(2N+1)/6 & -N(N+1)/2 \\ -N(N+1)/2 & N \end{pmatrix}$$

For large values of N we have

$$\hat{P}_N \approx \begin{pmatrix} 1/N & -6/N^2 \\ -6/N^2 & 12/N^3 \end{pmatrix}$$

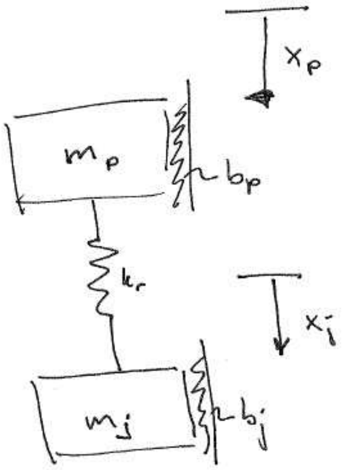
The diagonal elements describe the variance of the parameters a and b. The variance for the parameter decays as N^{-1} and the variance of the estimate of b decay as $12N^{-3}$. Which is to be shown.

Exercise 3



No conflict in the rules of causality, i.e. it is possible to describe the system as system of ordinary first order differential equations.

Exercise 4



$$\ddot{x}_p = \frac{1}{m_p}\left(-b_p\dot{x}_p + k_r(x_j - x_p) + m_p \cdot g\right)$$

$$\ddot{x}_j = \frac{1}{m_j}\left(-b_j\dot{x}_j - k_r(x_j - x_p) + m_j g\right)$$

$x_1 = x_p \quad x_2 = \dot{x}_p \quad x_3 = x_j \quad x_4 = \dot{x}_j$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \frac{1}{m_p}\left(-b_p x_2 + k_r(x_3 - x_1) + m_p g\right)$$
$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{1}{m_j}\left(-b_j x_4 - k_r(x_3 - x_1) + m_j g\right)$$

Exercise 5

(a)

a) $\dot{x} = -2x$

R-K: $x_{n+1} = x_n + hk_2$
$k_2 = f(x_n + \frac{h}{2}k_1)$
$k_1 = f(x_n)$

$k_1 = -2x_n$
$k_2 = f(x_n + \frac{h}{2}k_1) = f(x_n + \frac{h}{2}(-2x_n)) = f(x_n(1-h)) \cdot$
$= -2x_n(1-h)$

$x_{n+1} = x_n - 2hx_n(1-h) = (1 - 2h(1-h))x_n$

Stable if

$|1 - 2h(1-h)| < 1 \qquad h > 0$

$-1 < 1 - 2h(1-h) < 1$

$1 - 2h(1-h) > -1 \qquad 1 - 2h(1-h) < 1$
$-2h(1-h) > -2 \qquad -2h(1-h) < 0$
$h(1-h) < 1 \qquad 1 - h > 0$
$\boxed{h < 1} \qquad \boxed{h < 1}$

$\boxed{0 < h < 1}$

(b)

$x_1 = x(0,2) = x_0(1 - 2 \cdot 0,2(1-0,2)) =$
$= 1(1 - 0,4(0,8)) = 0,68$

$x_2 = 0,68(1 - 2 \cdot 0,2(1-0,2)) = 0,68 \cdot 0,68 = 0,68^2$