ESS100 Modelling and simulation
Solutions to exam Mon, 18 December 2006

Exercise 1

(a) It gives faster simulations.

(b) From the step response can the static gain and the time constant be determined. Static gain is given as b/a and the time constant as $1/a$.

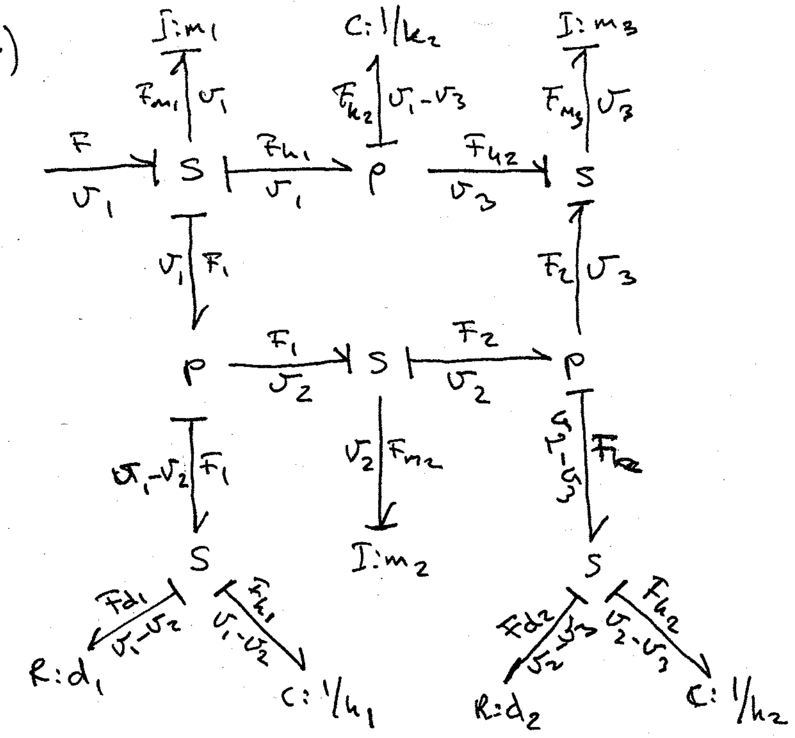
(c) The output signal does not depend on future values of the input signal, or in bond graph modelling, it is possible to write the system as an ODE..

(d) Test of the model. Examples are: cross validation (test of the model by use of a new measurement series), residuals and poles and zeros, frequency response (compare with spectral analysis)

(e) Make two different step responses with different magnitude of the step. Compare time constant and static gain, they should be independent on the steps magnitude.

2

a)



b) State variables: $v_1, v_2, v_3, F_{k_1}, F_{k_2}, F_{k_3}$

$$\dot v_1 = \frac{1}{m_1} F_{m_1} = \frac{1}{m_1}\left\{F - F_{k_2} - F_1\right\} = \frac{1}{m_1}\left\{F - F_{k_2} - F_{k_1} - d_1(v_1 - v_3)\right\}$$

$$\dot v_2 = \frac{1}{m_2} F_{m_2} = \frac{1}{m_1}\left\{F_1 - F_2\right\} = \frac{1}{m_2}\left\{F_{k_1} + d_1(v_1 - v_2) - F_{k_3} - d_2(v_2 - v_3)\right\}$$

$$\dot v_3 = \frac{1}{m_3} F_{m_3} = \frac{1}{m_3}\left\{F_{k_2} + F_2\right\} = \frac{1}{m_3}\left\{F_{k_2} + F_{k_3} + d_2(v_2 - v_3)\right\}$$

$$\dot F_{k_1} = k_1(v_1 - v_2)$$

$$\dot F_{k_2} = k_2(v_1 - v_3)$$

$$\dot F_{k_3} = k_3(v_2 - v_3)$$

$$\begin{bmatrix} \dot v_1 \\ \dot v_2 \\ \dot v_3 \\ \dot F_{k_1} \\ \dot F_{k_2} \\ \dot F_{k_3} \end{bmatrix} = \begin{bmatrix} -\frac{d_1}{m_1} & 0 & \frac{d_1}{m_1} & \cancel{0}\frac{1}{m_1} & -\frac{1}{m_1} & 0 \\ \frac{d_1}{m_2} & -\frac{(d_1+d_2)}{m_2} & \frac{d_2}{m_2} & \frac{1}{m_2} & 0 & -\frac{1}{m_2} \\ 0 & \frac{d_2}{m_3} & -\frac{d_2}{m_3} & 0 & \frac{1}{m_3} & \frac{1}{m_3} \\ k_1 & -k_1 & 0 & 0 & 0 & 0 \\ k_2 & 0 & -k_3 & 0 & 0 & 0 \\ 0 & k_3 & -k_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ F_{k_1} \\ F_{k_2} \\ F_{k_3} \end{bmatrix} + \begin{bmatrix} 1/m_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F$$

3)

$$M = \frac{K}{1+sT} U$$

$$M + T\dot{M} = KU$$

$$\dot{M} = -\frac{1}{T} M + \frac{K}{T} u$$

Select state variables and include actuator dynamics

$$[\theta_p, \dot{\theta}_p, \theta_a, \dot{\theta}_a, M] = [x_1 \; x_2 \; x_3 \; x_4 \; x_5]$$

a)

$$\begin{cases} \dot{\theta}_p = \dot{\theta}_p \\ \ddot{\theta}_p = \frac{1}{J_p}(mgl\sin\theta_p + \alpha\cos\theta_p) \\ \dot{\theta}_a = \dot{\theta}_a \\ \ddot{\theta}_a = \alpha \frac{M}{J_a} \\ \dot{M} = -\frac{1}{T} M + \frac{K}{T} u \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{1}{J_p}(mgl\sin x_1 + \alpha\cos x_1 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \frac{\alpha}{J_a} x_5 \\ \dot{x}_5 = -\frac{1}{T} x_5 + \frac{K}{T} u \end{cases}$$

b) Linearize

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{mgl}{J_p} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\alpha}{J_a} \\ 0 & 0 & 0 & 0 & -\frac{1}{T} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{K}{T} \end{bmatrix} u$$

1)

$$\hat{y}(t|\theta) = \theta^T \varphi(t) = [a_1 \;\; b_1] \begin{bmatrix} -y(t-1) \\ u(t-1) \end{bmatrix}$$

ARX model $\rightarrow$ *least squares*

$$\hat{\theta}_N = \left(\frac{1}{N}\sum_{t=1}^{N} \begin{bmatrix} -y(t-1) \\ u(t-1) \end{bmatrix} \begin{bmatrix} -y(t-1) & u(t-1) \end{bmatrix} \right)^{-1} \left(\frac{1}{N}\sum_{t=1}^{N} \begin{bmatrix} -y(t-1) \\ u(t-1) \end{bmatrix} y(t) \right) =$$

$$= \begin{bmatrix} \frac{1}{N}\sum_{t=1}^{N} y^2(t-1) & \frac{1}{N}\sum_{t=1}^{N} -y(t-1)u(t-1) \\ \frac{1}{N}\sum_{t=1}^{N} -y(t-1)u(t-1) & \frac{1}{N}\sum_{t=1}^{N} u^2(t-1) \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{N}\sum_{t=1}^{N} -y(t-1)y(t) \\ \frac{1}{N}\sum_{t=1}^{N} u(t-1)y(t) \end{bmatrix}$$

$$N \rightarrow \infty \;\; \Rightarrow \;\; \frac{1}{N}\sum_{1}^{N} \rightarrow E\{\;\}$$

assume stationary

$$E\{y^2(t-1)\} = E\{y^2(t)\} = E\{(0{,}7\,u(t-1) + 0{,}3\,u(t-2) + v(t))^2\}$$

$$= 0{,}49\,R_u(0) + 0{,}09\,R_u(0) + R_v(0) = 0{,}58 + 2 = 2{,}58$$

$$E\{y(t-1)u(t-1)\} = 0$$

$$E\{u^2(t-1)\} = R_u(0) = 1$$

$$E\{y(t-1)y(t)\} = E\{y(t-1)(0{,}7\,u(t-1) + 0{,}3\,u(t-2) + v(t))\} =$$

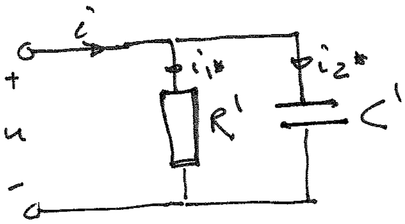
$$= E\{0{,}3\,y(t-1)u(t-2)\} = 0{,}3\,E\{u(t-2)(0{,}7\,u(t-2) + 0{,}3\,u(t-3) + v(t-1))\}$$

$$= 0{,}3 \cdot 0{,}7\,R_u(0) = 0{,}21$$

$$E\{u(t-1)y(t)\} = E\{u(t-1)(0{,}7\,u(t-1) + 0{,}3\,u(t-2) + v(t))\} = 0{,}7\,R_u(0) = 0{,}7$$

$$\hat{\theta} \rightarrow \begin{bmatrix} 2{,}58 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0{,}21 \\ 0{,}7 \end{bmatrix} = \begin{bmatrix} 0{,}21/2{,}58 \\ 0{,}7 \end{bmatrix}$$

a) parallel coupling of two resistors and two capicitors



$$R' = \frac{1}{\frac{1}{R_1} + \frac{1}{R}} = \frac{R}{2} \qquad C' = C + C = 2C$$

Kirchoffs current law : $i = i_{1*} + i_{2*}$

$$i_{1*} = \frac{u}{R'} \qquad i_{2*} = C'\frac{du}{dt}$$

This results in

$$C'\frac{du}{dt} + \frac{u}{R'} = i \quad \Rightarrow \quad \underline{\underline{\frac{du}{dt} = -\frac{u}{R'C'} + \frac{1}{C'}i}}$$

b) Differentiate one time

$$\left.\begin{aligned} C\frac{d^2u_1}{dt^2} + \frac{1}{R}\frac{du_1}{dt} - \frac{di_1}{dt} &= 0 \\ C\frac{d^2u_2}{dt^2} + \frac{1}{R}\frac{du_2}{dt} - \frac{di_2}{dt} &= 0 \\ \frac{du_1}{dt} - \frac{du_2}{dt} &= 0 \\ \frac{di_1}{dt} + \frac{di_2}{dt} - \frac{di}{dt} &= 0 \end{aligned}\right\}$$

Not enough to solve for $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$

Differentiate one more time gives

$$\frac{d^2u_1}{dt^2} - \frac{d^2u_2}{dt^2} = 0$$

Using this in combination with the equations above yields

$$\frac{di_1}{dt} - \frac{di_2}{dt} = 0 \quad \Rightarrow \quad \frac{di_1}{dt} = \frac{1}{2}\frac{di}{dt} \quad \frac{di_2}{dt} = \frac{1}{2}\frac{di}{dt}$$

$\frac{du_1}{dt}$ and $\frac{du_2}{dt}$ is already in ode form.

Index $= 2$

Exercise 6

(a)

$$\dot{x} = \begin{bmatrix} -3 & 10 \\ 0 & -1 \end{bmatrix} x = Ax$$

(b)

Euler method: $x_{n+1} = x_n + hf(x) = x_n + hAx$

A is on diagonal form => eigenvalues on the diagonal, λ_i. For stable simulation

$$x_{n+1} = x_n + h\lambda_i x_n = (1 + h\lambda_i)^n x_0$$

The simulation is stable if $|1 + h\lambda_i| < 1$

$\lambda_1 = -1$:

$$-1 < (1 - h) < 1$$
$$0 < h < 2$$

$\lambda_1 = -3$:

$$-1 < (1 - 3h) < 1$$
$$0 < h < 2/3$$

Stable if $h < 2/3$